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1996 J. Phys. A: Math. Gen. 29 2705

(http://iopscience.iop.org/0305-4470/29/11/010)

J. Phys. A: Math. Gen. **29** (1996) 2705–2735. Printed in the UK

Non-commutative geometry of finite groups

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Received 8 September 1995

Abstract. A finite set can be supplied with a group structure which can then be used to select (classes of) differential calculi on it via the notions of left-, right- and bicovariance. A corresponding framework has been developed by Woronowicz, more generally for Hopf algebras including quantum groups. A differential calculus is regarded as the most basic structure needed for the introduction of further geometric notions like linear connections and, moreover, for the formulation of field theories and dynamics on finite sets. Associated with each bicovariant first-order differential calculus on a finite group is a braid operator which plays an important role for the construction of distinguished geometric structures. For a covariant calculus, there are notions of invariance for linear connections and tensors. All these concepts are explored for finite groups and illustrated with examples. Some results are formulated more generally for arbitrary associative (Hopf) algebras. In particular, the problem of extension of a connection on a bimodule (over an associative algebra) to tensor products is investigated, leading to the class of 'extensible connections'. It is shown that invariance properties of an extensible connection on a bimodule over a Hopf algebra are carried over to the extension. Furthermore, an invariance property of a connection is also shared by a 'dual connection' which exists on the dual bimodule (as defined in this work).

1. Introduction

Non-commutative geometry (see [1], for example) replaces the familiar arena of classical physics, a manifold supplied with differential geometric structures, by an associative algebra $\mathcal{A}$ and algebraic structures on it. According to our point of view, the most basic geometric structure in the framework of non-commutative geometry is a 'differential calculus' on $\mathcal{A}$ (see also [2]). It allows the introduction of further geometric notions like linear connections and, moreover, the formulation of field theories and dynamics on finite sets.

Though non-commutative geometry is designed to handle non-commutative algebras, non-trivial structures already arise on commutative algebras with non-standard differential calculi (see [3] and references given therein). A commutative algebra of particular interest in this context is the algebra of functions on a finite (or discrete) set. A differential calculus on a finite set provides the latter with a structure which may be viewed as a discrete counterpart to that of a (continuous) differentiable manifold [4, 5]. It has been shown in [4] that (first-order) differential calculi on discrete sets are in correspondence with (di)graphs with at most two (antiparallel) arrows between any two vertices. This relation with graphs and networks suggests applications of the formalism to dynamics on networks [5] and the universal dynamics considered in [6], for example.

0305-4470/96/112705+31$19.50

A finite set can always be supplied with a group structure. The left and right action of the group on itself can then be used to distinguish certain differential calculi and geometric structures built on it [7–9]. A finite group together with a (bicovariant) differential calculus may be regarded as a 'finite Lie group'. The purpose of the present paper is to develop differential geometry on such spaces. In discrete (field) theories, discrete groups may appear as gauge groups, as isometry groups, and as structures underlying discrete spacetime models. For example, the hypercubic lattice underlying ordinary lattice (gauge) theories can be regarded as the abelian group $\mathbb{Z}^n$ (respectively, $\mathbb{Z}_N^n$ with a positive integer N, for a finite lattice). Lattice gauge theory can be understood as gauge theory on this group with a bicovariant differential calculus [10]. Another example which fits into our framework is the two-point space used in [11] to geometrize models of particle physics (see also [8, 9, 12]). In this model the group $\mathbb{Z}_2$ appears with the universal differential calculus (which is bicovariant).

Section 2 introduces differential calculus on finite groups and recalls the notions of left-, right- and bicovariance (using the language of Hopf algebras). For each bicovariant first-order differential calculus on a finite group and, more generally on a Hopf algebra, there is an operator which acts on the tensor product of 1-forms and satisfies the braid relation [7]. For a commutative finite group this is simply the permutation operator, but less trivial structures arise in the case of non-commutative groups. The generalized permutation operator can be used to define symmetric and antisymmetric tensor fields. All this is the subject of section 3. Linear connections on finite groups and corresponding invariance conditions are considered in section 4. Of particular interest are linear connections which can be extended to tensor products of 1-forms. We explore the restrictions on linear connections which arise from the extension property. Appendix A more generally addresses the problem of extending connections on two $\mathcal{A}$-bimodules, with $\mathcal{A}$ any associative algebra, to a connection on their tensor product. In section 5 we introduce vector fields on finite groups and briefly discuss a possible concept of a metric. In order to formulate, for example, metric-compatibility of a linear connection, the concept of the 'dual' of a connection is needed and the problem of its extensibility (in the sense mentioned above) has to be clarified. This is done in appendix B for an arbitrary associative algebra $\mathcal{A}$ and a connection on an $\mathcal{A}$-bimodule. An example of a non-commutative finite group is elaborated in section 6. Section 7 contains further discussion and conclusions. Appendix C recalls how coactions on two bimodules extend to a coaction on their tensor product. It is then shown that extensions of invariant connections are again invariant connections and that invariance is also carried over to the dual of a connection. In appendix D we briefly explore the concept of a 'two-sided connection' on a bimodule. Appendix E deals with invariant tensor fields on finite groups. Although in this work we concentrate on the case of finite sets supplied with a group structure, in appendix F we indicate how the formalism can be extended to the more general case of a finite set with a finite group acting on it.

Though originated from the development of 'differential geometry' on finite groups, some of our results are more general, they apply to arbitrary associative algebras, respectively, Hopf algebras. We therefore decided to separate them from the main part of the paper and placed them into a series of appendices (A–D).

2. Differential calculi on finite groups

Every finite set can be supplied with a group structure. If the number N of elements is prime, then the only irreducible group structure is $\mathbb{Z}_N$, the additive abelian group of integers modulo N. Differential calculi on discrete groups have been studied in [8, 9].

More generally, differential calculus on discrete sets has been developed in [4, 5].

Let $\mathcal{A}$ be the set of $\mathbb{C}$-valued functions on a finite set G. With each element $g \in G$ we associate a function $e_g \in \mathcal{A}$ via $e_g(g') = \delta_{g,g'}$. Then $e_g e_{g'} = \delta_{g,g'} e_g$ and $\sum_{g\in G} e_g = \mathbb{1}$ where $\mathbb{1}$ is the unit in $\mathcal{A}$. Every function f on G can be written as $f = \sum_{g\in G} f_g e_g$ with $f_g \in \mathbb{C}$. Choosing a group structure on G, the latter induces a *coproduct* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ via†

$$\Delta(f)(g, g') = f(gg'). \tag{2.1}$$

In particular,

$$\Delta(e_g) = \sum_{h\in G} e_h \otimes e_{h^{-1}g}. \tag{2.2}$$

A *differential calculus* on $\mathcal{A}$ is an extension of $\mathcal{A}$ to a differential algebra (Ω, d). Here $\Omega = \bigoplus_{r=0}^{\infty} \Omega^r$ is a graded associative algebra where $\Omega^0 = \mathcal{A}$. Ω^{r+1} is generated as an $\mathcal{A}$-bimodule via the action of a linear operator $\mathrm{d} : \Omega^r \to \Omega^{r+1}$ satisfying $\mathrm{d}^2 = 0$, $\mathrm{d}\mathbb{1} = 0$, and the graded Leibniz rule $\mathrm{d}(\varphi\varphi') = (\mathrm{d}\varphi)\varphi' + (-1)^r \varphi\, \mathrm{d}\varphi'$ where $\varphi \in \Omega^r$.‡

It is convenient [4, 9] to introduce the special 1-forms

$$e_{g,g'} := e_g \mathrm{d} e_{g'} \quad (g \neq g') \quad e_{g,g} := 0 \tag{2.3}$$

and the $(r-1)$-forms

$$e_{g_1,\ldots,g_r} := e_{g_1,g_2} e_{g_2,g_3} \ldots e_{g_{r-1},g_r}. \tag{2.4}$$

They satisfy

$$e_{g_1,\ldots,g_r} e_{h_1,\ldots,h_s} = \delta_{g_r,h_1} e_{g_1,\ldots,g_r,h_2,\ldots,h_s}. \tag{2.5}$$

The operator d acts on them as follows:

$$\mathrm{d}e_{g_1,\ldots,g_r} = \sum_{h\in G} [e_{h,g_1,\ldots,g_r} - e_{g_1,h,g_2,\ldots,g_r} + e_{g_1,g_2,h,g_3,\ldots,g_r} - \cdots + (-1)^r\, e_{g_1,\ldots,g_r,h}]. \tag{2.6}$$

If no further relations are imposed, one is dealing with the 'universal differential calculus' $(\tilde{\Omega}, \tilde{\mathrm{d}})$. The $e_{g_1,\ldots,g_r}$ with $g_i \neq g_{i+1}$ $(i = 1, \ldots, r-1)$ then constitute a basis over $\mathbb{C}$ of $\tilde{\Omega}^{r-1}$ for $r > 1$ [4]. Every other differential calculus on G is obtained from $\tilde{\Omega}$ as the quotient with respect to some two-sided differential ideal. Up to first order, i.e. the level of 1-forms, every differential calculus on G is obtained by setting some of the $e_{g,g'}$ to zero. Via (2.4) and (2.6) this induces relations for forms of higher grade. In addition, or alternatively, one may also factor out ideals generated by forms of higher grade. Every first-order differential calculus on G can be described by a (di)graph the vertices of which are the elements of G and there is an arrow pointing from a vertex g to a vertex g' iff $e_{g,g'} \neq 0$ (see [4] for further details).

A differential calculus on G (or, more generally, any Hopf algebra $\mathcal{A}$) is called *left-covariant* [7] if there is a linear map $\Delta_{\Omega^1} : \Omega^1 \to \mathcal{A} \otimes \Omega^1$ such that

$$\Delta_{\Omega^1}(f\varphi f') = \Delta(f)\Delta_{\Omega^1}(\varphi)\Delta(f') \qquad \forall f, f' \in \mathcal{A},\ \varphi \in \Omega^1 \tag{2.7}$$

and

$$\Delta_{\Omega^1} \circ \mathrm{d} = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta. \tag{2.8}$$

† Here we make use of the fact that a group defines a Hopf algebra. This formalism is adequate, in particular, if one has in mind to generalize the structures considered in the present work to non-commutative Hopf algebras like quantum groups.

‡ In [5] a discrete set together with a differential calculus on it has been called *discrete differential manifold*. This notion was motivated by a far-reaching analogy [4] with the continuum case where Ω is the algebra of differential forms on a manifold and d the exterior derivative.

As a consequence†,

$$\Delta_{\Omega^1}(e_{g,g'}) = \sum_{h\in G} e_{h^{-1}} \otimes e_{hg,hg'}. \tag{2.9}$$

Hence, in order to find the left-covariant differential calculi on G, we have to determine the orbits of all elements of $(G \times G)'$ where the prime indicates omission of the diagonal (i.e. $(G\times G)' = (G\times G)\setminus\{(g, g) \mid g \in G\})$ with respect to the left action $(g, g') \mapsto (hg, hg')$. In the graph picture, left-covariant first-order differential calculi are obtained from the universal one (which is left-covariant) by deleting corresponding orbits of arrows.

For a left-covariant‡ differential calculus, there are left-invariant 'Maurer–Cartan' 1-forms [8, 9]

$$\theta^g := \sum_{h\in G} e_{hg,h} = e_{\underline{h}g,\underline{h}} = e_{\underline{h},\underline{h}g^{-1}} \qquad \Delta_{\Omega^1}(\theta^g) = \mathbb{1} \otimes \theta^g. \tag{2.10}$$

Here we have introduced a summation convention. If an index is underlined, this means summation over all group elements. Note that $\theta^e = 0$ according to the above definition. Furthermore, the Maurer–Cartan forms with $g \neq e$ are in one-to-one correspondence with left orbits in $(G \times G)'$. All left-covariant differential calculi (besides the universal one) are therefore obtained by setting some of the θ^g (of the universal calculus) to zero. The non-vanishing θ^g then constitute a left (or right) $\mathcal{A}$-module basis for Ω^1 since

$$e_{h,g} = e_h\theta^{g^{-1}h} = \theta^{g^{-1}h}e_g. \tag{2.11}$$

As a generalization of the last equality we have the simple commutation relations

$$f\theta^g = \theta^g\mathcal{R}_g f \tag{2.12}$$

where $\mathcal{R}_g$ denotes the action of G on $\mathcal{A}$ induced by right multiplication, i.e.

$$(\mathcal{R}_g f)(h) := f(hg) \ (\forall f \in \mathcal{A}) \qquad \mathcal{R}_g\mathcal{R}_h = \mathcal{R}_{gh}. \tag{2.13}$$

The equation (2.11) can be used to prove the Maurer–Cartan equations

$$\mathrm{d}\theta^h = -C^h{}_{\underline{g},\underline{g'}}\theta^{\underline{g'}}\theta^{\underline{g}} \tag{2.14}$$

with the 'structure constants'

$$C^h{}_{g,g'} := -\delta^h_g - \delta^h_{g'} + \delta^h_{gg'}. \tag{2.15}$$

These have the property

$$C^{\mathrm{ad}(h)g}{}_{\mathrm{ad}(h)g',\mathrm{ad}(h)g''} = C^g{}_{g',g''} \qquad \forall h \in G \tag{2.16}$$

where ad denotes the adjoint action of G on G, i.e. $\mathrm{ad}(h)g = hgh^{-1}$.

A differential calculus on G is called *right-covariant* if there is a linear map ${}_{\Omega^1}\Delta$: $\Omega^1 \to \Omega^1 \otimes \mathcal{A}$ such that

$${}_{\Omega^1}\Delta(f\varphi f') = \Delta(f){}_{\Omega^1}\Delta(\varphi)\Delta(f') \qquad {}_{\Omega^1}\Delta \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta. \tag{2.17}$$

This implies

$${}_{\Omega^1}\Delta(e_{g,g'}) = \sum_{h\in G} e_{gh,g'h} \otimes e_{h^{-1}}. \tag{2.18}$$

† Without refering to the Hopf-algebraic language, left-covariance of a differential calculus basically means $\mathcal{L}_g\mathrm{d} = \mathrm{d}\mathcal{L}_g$ where $\mathcal{L}_g$ is the action on $\mathcal{A}$ induced by the left multiplication of elements of G by g, see (2.23). Then $\mathcal{L}_g e_{h,h'} = e_{g^{-1}h,g^{-1}h'}$ which corresponds to (2.9).

‡ All following formulae and statements which do not make explicit reference to a coaction are actually valid without the assumption of left-covariance. The second formula in (2.10) is based on it, however, and it is this invariance property which justifies our definition in (2.10).

For a right-covariant differential calculus, there are right-invariant Maurer–Cartan 1-forms,

$$\omega^g := e_{g\underline{h},\underline{h}} \qquad {}_{\Omega^1}\Delta(\omega^g) = \omega^g \otimes \mathbb{1} \tag{2.19}$$

which satisfy

$$e_{h,g} = e_h \omega^{hg^{-1}} = \omega^{hg^{-1}} e_g \tag{2.20}$$

$$\mathrm{d}\omega^h = -C^h{}_{\underline{g},\underline{g}'} \omega^{\underline{g}} \omega^{\underline{g}'} \tag{2.21}$$

and

$$f \omega^g = \omega^g \mathcal{L}_g f \tag{2.22}$$

where $\mathcal{L}_g$ denotes the action of G on $\mathcal{A}$ induced by left multiplication,

$$(\mathcal{L}_g f)(h) := f(gh) \ (\forall f \in \mathcal{A}) \qquad \mathcal{L}_g \mathcal{L}_h = \mathcal{L}_{hg}. \tag{2.23}$$

A differential calculus is *bicovariant* if it is left- and right-covariant. Then, in the case under consideration,

$$_{\Omega^1}\Delta(\theta^g) = \theta^{\mathrm{ad}(\underline{h})g} \otimes e_{\underline{h}}. \tag{2.24}$$

It follows that bicovariant calculi are in one-to-one correspondence with unions of conjugacy classes different from $\{e\}$. Obviously,

$$\rho := \theta^{\underline{g}} = \omega^{\underline{g}} = e_{\underline{g},\underline{g}'} \tag{2.25}$$

is a *bi-invariant* 1-form.

In the following we list some useful formulae. For $f \in \mathcal{A}$ we find

$$\mathrm{d}f = [\rho, f] = (\ell_{\underline{g}} f)\theta^{\underline{g}} = (r_{\underline{g}} f)\omega^{\underline{g}} \tag{2.26}$$

where

$$\ell_g f := \mathcal{R}_{g^{-1}} f - f \qquad r_g f := \mathcal{L}_{g^{-1}} f - f. \tag{2.27}$$

Using (2.23) and (2.13), it is easy to check that

$$\ell_g \ell_{g'} = C^{\underline{h}}{}_{g',g} \ell_{\underline{h}} \qquad r_g r_{g'} = C^{\underline{h}}{}_{g,g'} r_{\underline{h}}. \tag{2.28}$$

The 1-forms θ^g and ω^g are related as follows:

$$\theta^g = e_{\underline{h}} \omega^{\mathrm{ad}(\underline{h})g} \qquad \omega^g = e_{\underline{h}} \theta^{\mathrm{ad}(\underline{h}^{-1})g}. \tag{2.29}$$

In the following sections we restrict our considerations to differential calculi which are at least left-covariant. As already mentioned, in this case the set of non-vanishing left-invariant Maurer–Cartan 1-forms θ^g is a basis of Ω^1 as a left $\mathcal{A}$-module. It is then convenient to introduce the subset $\hat{G} := \{g \in G | \theta^g \neq 0\}$ of G. If not said otherwise, indices will be restricted to $\hat{G}$ in what follows. This does *not* apply to our summation convention, however. Underlining an index still means summation over all elements of G, though in most cases the sum reduces to a sum over $\hat{G}$ (but see (3.2) for an exception).

3. The canonical bimodule isomorphism for a bicovariant differential calculus

For a bicovariant differential calculus there is a unique bimodule isomorphism $\sigma : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ such that

$$\sigma(\theta \otimes_{\mathcal{A}} \omega) = \omega \otimes_{\mathcal{A}} \theta \tag{3.1}$$

for all left-invariant 1-forms θ and right-invariant 1-forms ω [7]. We have $\sigma(\rho \otimes_{\mathcal{A}} \rho) = \rho \otimes_{\mathcal{A}} \rho$ since ρ is bi-invariant. Furthermore,

$$\begin{aligned}\sigma(\theta^g \otimes_{\mathcal{A}} \theta^{g'}) &= \sigma(\theta^g \otimes_{\mathcal{A}} e_{\underline{h}} \omega^{\mathrm{ad}(\underline{h})g'}) \\ &= \sigma(\theta^g e_{\underline{h}} \otimes_{\mathcal{A}} \omega^{\mathrm{ad}(\underline{h})g'}) \\ &= e_{\underline{h}g} \sigma(\theta^g \otimes_{\mathcal{A}} \omega^{\mathrm{ad}(\underline{h})g'}) \\ &= e_{\underline{h}g} \omega^{\mathrm{ad}(\underline{h})g'} \otimes_{\mathcal{A}} \theta^g \\ &= e_{\underline{h}g} e_{\underline{h}'} \theta^{\mathrm{ad}(\underline{h}'^{-1}\underline{h})g'} \otimes_{\mathcal{A}} \theta^g \\ &= e_{\underline{h}} \theta^{\mathrm{ad}(g^{-1})g'} \otimes_{\mathcal{A}} \theta^g \end{aligned} \tag{3.2}$$

which implies

$$\sigma(\theta^g \otimes_{\mathcal{A}} \theta^{g'}) = \theta^{\mathrm{ad}(g^{-1})g'} \otimes_{\mathcal{A}} \theta^g. \tag{3.3}$$

In particular, $\sigma(\theta^g \otimes_{\mathcal{A}} \theta^g) = \theta^g \otimes_{\mathcal{A}} \theta^g$. More generally, it is possible to calculate an expression for higher powers of σ. By induction one can prove that

$$\sigma^{2n-1}(\theta^g \otimes_{\mathcal{A}} \theta^h) = \theta^{\mathrm{ad}(g^{-1}h^{-1})^n h} \otimes_{\mathcal{A}} \theta^{\mathrm{ad}(g^{-1}h^{-1})^{n-1} g} \tag{3.4}$$

$$\sigma^{2n}(\theta^g \otimes_{\mathcal{A}} \theta^h) = \theta^{\mathrm{ad}(g^{-1}h^{-1})^n g} \otimes_{\mathcal{A}} \theta^{\mathrm{ad}(g^{-1}h^{-1})^n h} \tag{3.5}$$

for all $n \geqslant 1$. With the help of the last formula one arrives at the following result.

Proposition 3.1. For a finite group G and a bicovariant first-order differential calculus on it, the associated bimodule isomorphism σ satisfies

$$\sigma^{2|\mathrm{ad}(G)|} = \mathrm{id} \tag{3.6}$$

where $|\mathrm{ad}(G)|$ denotes the number of elements of $\mathrm{ad}(G) := \{\mathrm{ad}(g) | g \in G\}$, the group of inner automorphisms of G.

Proof. For $a \in \mathrm{ad}(G)$ let $\langle a \rangle$ denote the cyclic subgroup of $\mathrm{ad}(G)$ generated by a. Since $\mathrm{ad}(G)$ is a finite group, $|\langle a \rangle|$ is finite and $a^{|\langle a \rangle|} = \mathrm{id}$. Furthermore, $|\langle a \rangle|$ is a divisor of $|\mathrm{ad}(G)|$ by Lagrange's theorem. Now (3.6) follows from (3.5). □

We define the *order* of σ as the smallest positive integer m such that $\sigma^m = \mathrm{id}$ and denote it as $|\sigma|$. The previous proposition then tells us that $|\sigma| \leqslant 2|\mathrm{ad}(G)|$. Our next result shows that, in general, equality does not hold. For the symmetric groups $\mathcal{S}_n$ with $n > 3$ one finds that $|\sigma| < 2|\mathrm{ad}(\mathcal{S}_n)|$.

Proposition 3.2. For the symmetric group $\mathcal{S}_n$, $n \geqslant 3$, with the universal first-order differential calculus, we have†

$$|\sigma| = 2n \prod_{k=1}^{n-2} \frac{n-k}{\gcd[n(n-1)\ldots(n-k+1), n-k]} \tag{3.7}$$

where $\gcd[\ell, \ell']$ denotes the greatest common divisor of positive integers ℓ and ℓ'.

† Another way to describe the number on the right-hand side is the following. Write all the factors $2, \ldots, n$ of $|\mathcal{S}_n| = n!$ as products of powers of primes. Then $|\sigma|$ is twice the product of all different primes, each taken to the power which is the highest with which the prime appears in the set of factors $2, \ldots, n$.

Proof. For $n > 2$ the centre of $\mathcal{S}_n$ is trivial [13] and the group of inner automorphisms is therefore isomorphic with $\mathcal{S}_n$ itself. Every element $g \in \mathcal{S}_n$, $g \neq e$, can be written as a product of disjoint (and thus commuting) cycles. Hence $g^\ell = e$ with $\ell := n\prod_{k=1}^{n-2}(n-k)/\mathrm{gcd}[n(n-1)\ldots(n-k+1), n-k]$. By construction, ℓ is the smallest positive integer with the property $g^\ell = e$ for all $g \in \mathcal{S}_n$, since for each divisor of ℓ there is a cyclic subgroup of order equal to this divisor in $\mathcal{S}_n$ (given by a cycle of length equal to the divisor). For the universal differential calculus on $\mathcal{S}_n$, the statement in the proposition now follows from (3.4) and (3.5). □

For a bicovariant differential calculus, it is natural to consider the following symmetrization and antisymmetrization operators acting on $\Omega^1 \otimes_\mathcal{A} \Omega^1$,

$$\boldsymbol{S} := \tfrac{1}{2}(\mathrm{id} + \sigma) \qquad \boldsymbol{A} := \tfrac{1}{2}(\mathrm{id} - \sigma). \tag{3.8}$$

In general, $\sigma^2 \neq \mathrm{id}$, so that these are not projections. It is therefore not quite straightforward how to define symmetry and antisymmetry for an element $\alpha \in \Omega^1 \otimes_\mathcal{A} \Omega^1$. We suggest the following notions,

α is w-symmetric iff $\quad \alpha \in \boldsymbol{S}(\Omega^1 \otimes_\mathcal{A} \Omega^1) = \mathrm{im}\,\boldsymbol{S}$

α is s-symmetric iff $\quad \boldsymbol{S}(\alpha) = \alpha$

α is w-antisymmetric iff $\quad \alpha \in \boldsymbol{A}(\Omega^1 \otimes_\mathcal{A} \Omega^1) = \mathrm{im}\,\boldsymbol{A}$

α is s-antisymmetric iff $\quad \boldsymbol{A}(\alpha) = \alpha$

where ‘w’ and ‘s’ stand for ‘weakly’ and ‘strongly’, respectively†. Examples are treated in section 6.1 and appendix E. The notions of s-symmetry and w-antisymmetry are complementary in the following sense.

Proposition 3.3. For each bicovariant differential calculus Ω on a finite group G, the space $\Omega^1 \otimes_\mathcal{A} \Omega^1$ decomposes into direct sums

$$\Omega^1 \otimes_\mathcal{A} \Omega^1 = \ker \boldsymbol{A} \oplus \mathrm{im}\,\boldsymbol{A} = \ker \boldsymbol{S} \oplus \mathrm{im}\,\boldsymbol{S}. \tag{3.9}$$

Proof. In order to show the first equality, it is sufficient to prove that $\ker \boldsymbol{A} \cap \mathrm{im}\,\boldsymbol{A} = 0$ since $\Omega^1 \otimes_\mathcal{A} \Omega^1$ is a finite-dimensional vector space over $\mathbb{C}$ and $\boldsymbol{A}$ a linear map. Let α be an element of $\ker \boldsymbol{A} \cap \mathrm{im}\,\boldsymbol{A}$. Then $\sigma(\alpha) = \alpha$ and $\alpha = (\sigma - \mathrm{id})(\beta)$ with an element $\beta \in \boldsymbol{A}(\Omega^1 \otimes_\mathcal{A} \Omega^1)$. Using (3.6) we obtain

$$0 = (\sigma^{2|\mathrm{ad}(G)|} - \mathrm{id})(\beta) = \left(\sum_{k=0}^{2|\mathrm{ad}(G)|-1} \sigma^k\right)(\sigma - \mathrm{id})(\beta) = \left(\sum_{k=0}^{2|\mathrm{ad}(G)|-1} \sigma^k\right)(\alpha) = 2|\mathrm{ad}(G)|\alpha.$$

Hence, $\alpha = 0$. In the same way, the second equality in (3.9) is verified with the help of $0 = (\sum_{k=0}^{2|\mathrm{ad}(G)|-1}(-1)^{k+1}\sigma^k)(\sigma + \mathrm{id})$. □

We note that σ satisfies the braid equation‡

$$(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) \tag{3.10}$$

(see also [7]). In [7] Woronowicz implemented a generalized wedge product by taking the quotient of $\Omega^1 \otimes_\mathcal{A} \Omega^1$ with respect to the subbimodule of s-symmetric tensors, i.e.

† The conditions $\boldsymbol{S}(\alpha) = \alpha$ and $\boldsymbol{A}(\alpha) = \alpha$ are equivalent to $\alpha \in \ker \boldsymbol{A}$ and $\alpha \in \ker \boldsymbol{S}$, respectively.

‡ The fact that σ satisfies the braid relation has the following origin. Let V be a vector space and Φ a map $V \to \mathrm{End}(V)$. The map $\tilde{\sigma} : V \otimes V \to V \otimes V$ defined by $\tilde{\sigma}(x \otimes y) := y \otimes \Phi_y x$ then satisfies the braid equation if and only if $\Phi_x \circ \Phi_y = \Phi_{\Phi_x y} \circ \Phi_x$ for all $x, y \in V$. In particular, if V is the group algebra of a (not necessarily finite) group G, then $\Phi = \mathrm{ad}$ satisfies this equation.

$\Omega^2 = (\Omega^1 \otimes_{\mathcal{A}} \Omega^1)/\ker \boldsymbol{A}$ which can be identified with the space of w-antisymmetric tensors†. Then $\mathrm{d}\rho = \rho^2 = 0$.

Example. We consider a set of three elements with the group structure $\mathbb{Z}_3$. The $\mathbb{Z}_3$ left-covariant first-order differential calculi on this set are then represented by the graphs in figure 1.

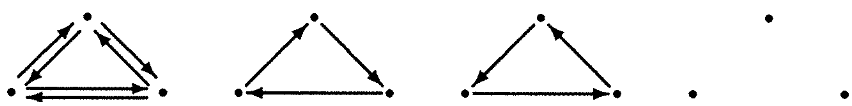

Figure 1. The digraphs which determine all left-covariant first-order differential calculi on $\mathbb{Z}_3$.

The last of these graphs has no arrows and corresponds to the trivial differential calculus (where $\mathrm{d} \equiv 0$). For a commutative group, as in the present example, bicovariance does not lead to additional conditions. Since left- and right-invariant Maurer–Cartan forms coincide for a commutative group, the map σ acts on left-invariant forms simply as permutation, i.e. $\sigma(\theta^g \otimes_{\mathcal{A}} \theta^{g'}) = \theta^{g'} \otimes_{\mathcal{A}} \theta^g$. In particular, $\sigma^2 = \mathrm{id}$ in accordance with proposition 3.1 and the wedge product determined by σ is therefore the ordinary one for left-invariant 1-forms, though we still do not have anticommutativity of the product of two 1-forms, in general.

More complicated maps σ with $\sigma^2 \neq \mathrm{id}$ arise from a non-commutative group structure. In section 6 we elaborate in some detail the case of the symmetric group $\mathcal{S}_3$.

4. Linear connections on a finite group

Let $\mathcal{A}$ be an associative algebra and Ω a differential calculus on it. A *connection* on a left $\mathcal{A}$-module Γ is a map‡

$$\nabla : \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma \tag{4.1}$$

such that

$$\nabla(f\gamma) = \mathrm{d}f \otimes_{\mathcal{A}} \gamma + f\nabla\gamma \qquad \forall f \in \mathcal{A},\ \gamma \in \Gamma \tag{4.2}$$

[15]. A connection on Γ can be extended to a map $\Omega \otimes_{\mathcal{A}} \Gamma \to \Omega \otimes_{\mathcal{A}} \Gamma$ via

$$\nabla(\varphi\Psi) = (\mathrm{d}\varphi)\Psi + (-1)^r \varphi\nabla\Psi \tag{4.3}$$

for $\varphi \in \Omega^r$ and $\Psi \in \Omega \otimes_{\mathcal{A}} \Gamma$. Then ∇^2, which is a left $\mathcal{A}$-module homomorphism, defines the *curvature* of the connection.

In the following we consider the particular case where $\Gamma = \Omega^1$, the space of 1-forms of a differential calculus on $\mathcal{A}$. A connection is then called a *linear connection*. It is a map

† Alternatively, one may think of implementing a generalized wedge product by taking the quotient with respect to w-symmetric tensors. However, this turns out to be too restrictive, in general (see section 6.1 and the example in appendix E). Moreover, one may also consider corresponding (anti)symmetry conditions obtained from those given above by replacing σ by some power of σ and use them to define a wedge product. These possibilities reflect the fact that there are several differential algebras with the same space of 1-forms. These define different discrete differential manifolds [4, 5]. The choice made by Woronowicz is uniquely distinguished by the property that bicovariance extends to the whole differential algebra, see theorem 4.1 in [7].

‡ Similarly, a connection on a right $\mathcal{A}$-module is a map $\nabla : \Gamma \to \Gamma \otimes_{\mathcal{A}} \Omega^1$ with $\nabla(\gamma f) = (\nabla\gamma) f + \gamma \otimes_{\mathcal{A}} \mathrm{d}f$. A left (right) module over an associative algebra $\mathcal{A}$ has a connection with respect to the universal first-order differential calculus if and only if it is projective [14] (see also [15]).

$\nabla : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$†. The *torsion* of a linear connection may be defined as

$$T = \mathrm{d} - \pi \circ \nabla \tag{4.4}$$

where π is the projection $\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^2$‡.

If Ω^1 has a left $\mathcal{A}$-module basis θ^i, $i = 1, \ldots, n$, the action of a linear connection on a 1-form $\varphi = \varphi_i \theta^i$ (summation convention) is given by

$$\nabla\varphi = D\varphi_i \otimes_{\mathcal{A}} \theta^i \tag{4.5}$$

where we have introduced

$$D\varphi_i := \mathrm{d}\varphi_i - \varphi_j \omega^j{}_i \tag{4.6}$$

with connection 1-forms

$$\omega^i{}_j = \Gamma^i{}_{jk}\theta^k \tag{4.7}$$

defined by

$$\nabla\theta^i = -\omega^i{}_j \otimes_{\mathcal{A}} \theta^j. \tag{4.8}$$

Extending a linear connection on Ω^1 to a connection on $\Omega \otimes_{\mathcal{A}} \Omega^1$,

$$\nabla^2\theta^i = -\Omega^i{}_j \otimes_{\mathcal{A}} \theta^j \tag{4.9}$$

defines curvature 2-forms§ for which we obtain the familiar formula

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \omega^k{}_j. \tag{4.10}$$

Remark. Under a change of basis $\theta^i \mapsto a^i{}_j \theta^j$ where a is an invertible matrix with entries in $\mathcal{A}$, we have the tensorial transformation properties $\varphi_i \mapsto \varphi_j (a^{-1})^j{}_i$ and $D\varphi_i \mapsto D\varphi_j (a^{-1})^j{}_i$. For the connection 1-forms and the curvature 2-forms one finds the familiar transformation laws $\omega^i{}_j \mapsto a^i{}_k \omega^k{}_l (a^{-1})^l{}_j + a^i{}_k \mathrm{d}(a^{-1})^k{}_j$ and $\Omega^i{}_j \mapsto a^i{}_k \Omega^k{}_l (a^{-1})^l{}_j$. It should be noticed, however, that the components $R^i{}_{jkl}$ of the 2-forms $\Omega^i{}_j$ with respect to the generators $\theta^k\theta^l$ of Ω^2 do *not* transform in this simple way if functions (here the entries of the transformation matrix a) do not commute with all 1-forms (here the basis 1-forms θ^k), as in the case of a differential calculus on a finite set‖.

Now we turn to the special case of a left-covariant differential calculus on a finite group. As a left $\mathcal{A}$-module basis we choose the set of left-invariant Maurer–Cartan 1-forms θ^g. Except where stated otherwise, indices are restricted to $\hat{G}$. For $\varphi = \varphi_{\underline{g}} \theta^{\underline{g}}$ one finds

$$\nabla\varphi = (\mathcal{R}_{\underline{g}^{-1}} \varphi_{\underline{g}'} - \varphi_{\underline{h}} U^{\underline{h}}{}_{\underline{g}',\underline{g}}) \theta^{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g}'} \tag{4.11}$$

where

$$U^h{}_{g',g} := \delta^h_{g'} + \Gamma^h{}_{g',g}. \tag{4.12}$$

As a consequence,

$$\nabla\varphi = 0 \Leftrightarrow \mathcal{R}_{g^{-1}} \varphi_{g'} = \varphi_{\underline{h}} U^{\underline{h}}{}_{g',g}. \tag{4.13}$$

† This pretends that there should be a kind of symmetry with respect to the two factors of the tensor product. From the general formula (4.1) it should be clear that the two factors play very different roles.

‡ For a bicovariant first-order differential calculus on a Hopf algebra, the choice made in [7] is $\pi = \boldsymbol{A}$, see section 3.

§ More precisely, these are the components of the curvature with respect to the (arbitrary) left $\mathcal{A}$-module basis θ^i.

‖ On the other hand, it is precisely such a change of the ordinary transformation law for components of forms which underlies the derivation of the lattice gauge theory action in [10].

Left-invariance of ∇ (see appendix C) is equivalent to $\Gamma^{g}_{g',g''} \in \mathbb{C}$. If the differential calculus is bicovariant, evaluation of the right-invariance condition (see appendix C) then leads to†

$$\Gamma^{\mathrm{ad}(h)g}_{\mathrm{ad}(h)g',\mathrm{ad}(h)g''} = \Gamma^{g}_{g',g''} \qquad \forall g, g', g'' \in \hat{G}, \forall h \in G. \tag{4.14}$$

For a left-invariant connection‡,

$$\Omega^{g}{}_{g'} = (\Gamma^{g}{}_{\underline{g''},\underline{h}}\Gamma^{\underline{g''}}{}_{g',\underline{h'}} - C^{\underline{g''}}{}_{\underline{h'},\underline{h}}\Gamma^{g}{}_{g',\underline{g''}})\theta^{\underline{h}}\theta^{\underline{h'}}. \tag{4.15}$$

Applying T to the left-invariant basis, we find

$$T(\theta^{h}) = (\Gamma^{h}{}_{\underline{g'},\underline{g}} - C^{h}{}_{\underline{g'},\underline{g}})\theta^{\underline{g}}\theta^{\underline{g'}}. \tag{4.16}$$

The constants $C^{h}{}_{g',g}$ are those defined in (2.15).

Example 1. For a bicovariant (first-order) differential calculus,

$$\nabla^{\sigma}\varphi := \rho \otimes_{\mathcal{A}} \varphi - \sigma(\varphi \otimes_{\mathcal{A}} \rho) \tag{4.17}$$

defines a linear left $\mathcal{A}$-module connection. Here σ is the canonical bimodule isomorphism. For this connection the Maurer–Cartan 1-forms θ^{g} are covariantly constant, i.e. $\nabla^{\sigma}\theta^{g} = 0$. As a consequence, the connection is bi-invariant, the curvature vanishes and the torsion is given by $T(\theta^{h}) = -C^{h}{}_{\underline{g'},\underline{g}}\theta^{\underline{g}}\theta^{\underline{g'}} = \mathrm{d}\theta^{h}$.

The connection (4.17) can be generalized to a family of bi-invariant§ left $\mathcal{A}$-module connections,

$$\nabla^{(\lambda_0,\ldots,\lambda_{|\sigma|-1})}\varphi := \rho \otimes_{\mathcal{A}} \varphi - \sum_{n=0}^{|\sigma|-1} \lambda_n \sigma^{n}(\varphi \otimes_{\mathcal{A}} \rho) \tag{4.18}$$

where $|\sigma|$ is the order of σ (see section 3) and $\lambda_n \in \mathbb{C}$‖. It includes $\nabla^{\sigma^{-1}} := \nabla^{(0,\ldots,0,1)}$. With respect to this connection the right-invariant Maurer–Cartan 1-forms ω^{g} are covariantly constant, i.e. $\nabla^{\sigma^{-1}}\omega^{g} = 0$, and the curvature also vanishes. The two connections ∇^{σ} and $\nabla^{\sigma^{-1}}$ provide us with analogues of the $(+)$- and $(-)$-parallelism on Lie groups (see [17], section 50). Corresponding *right* $\mathcal{A}$-module connections with these properties are given by $\sigma^{-1} \circ \nabla^{\sigma}$ and $\sigma \circ \nabla^{\sigma^{-1}}$.

Example 2. For a bicovariant differential calculus with the (generalized) wedge product as defined by Woronowicz (see section 3), the torsion and the curvature of a left-invariant linear (left $\mathcal{A}$-module) connection are given by

$$T(\theta^{h}) = \tfrac{1}{2}(\Gamma^{h}{}_{\underline{g},\underline{g'}} - \Gamma^{h}{}_{\mathrm{ad}(\underline{g})\underline{g'},\underline{g}} - C^{h}{}_{\underline{g},\underline{g'}} + C^{h}{}_{\mathrm{ad}(\underline{g})\underline{g'},\underline{g}})\theta^{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g'}} \tag{4.19}$$

$$\Omega^{g}{}_{g'} = \tfrac{1}{2}[\Gamma^{g}{}_{\underline{g''},\underline{h}}\Gamma^{\underline{g''}}{}_{g',\underline{h'}} - \Gamma^{g}{}_{\underline{g''},\underline{h'}}\Gamma^{\underline{g''}}{}_{g',\mathrm{ad}(\underline{h'})\underline{h}} + (C^{\underline{g''}}{}_{\mathrm{ad}(\underline{h'})\underline{h},\underline{h'}} - C^{\underline{g''}}{}_{\underline{h'},\underline{h}})\Gamma^{g}{}_{g',\underline{g''}}]\theta^{\underline{h}} \otimes_{\mathcal{A}} \theta^{\underline{h'}} \tag{4.20}$$

using $\Omega^{2} \cong \mathrm{im}\,\boldsymbol{A}$ (cf section 3) and (3.3). The condition of vanishing torsion for a linear connection is

$$\Gamma^{h}{}_{g,g'} - \Gamma^{h}{}_{\mathrm{ad}(g)g',g} = C^{h}{}_{g,g'} - C^{h}{}_{\mathrm{ad}(g)g',g} = -\delta^{h}_{g'} + \delta^{h}_{\mathrm{ad}(g)g'} \tag{4.21}$$

which, for a commutative group, reduces to $\Gamma^{h}{}_{g,g'} = \Gamma^{h}{}_{g',g}$.

† Note that, for a bicovariant differential calculus, $\mathrm{ad}(h)g \in \hat{G}$ whenever $g \in \hat{G}, h \in G$.

‡ For the universal differential calculus this implies $\Omega = 0 \Leftrightarrow U_g U_{g'} = U_{gg}$ where $U_g := (U^{h}{}_{h',g})$. The curvature thus measures the deviation of the matrices U_g from being a representation of the group G.

§ The 1-form ρ and the bimodule isomorphism σ are bi-invariant [7]. The bi-invariance of the connections (4.18) then follows from proposition C.3.

‖ More generally, if Ψ is any left $\mathcal{A}$-module homomorphism $\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1$, then $\nabla^{\Psi}\varphi := \rho \otimes_{\mathcal{A}} \varphi - \Psi(\varphi \otimes_{\mathcal{A}} \rho)$ is a linear connection, see also [16].

Example 3. For a left-covariant first-order differential calculus,

$$\Gamma^h{}_{g,g'} = C^h{}_{g,g'} \tag{4.22}$$

defines a left-invariant linear connection which we call the *C-connection*. Independent of the continuation of the first-order calculus to higher orders, for this connection the torsion vanishes. For a bicovariant differential calculus, the C-connection is bi-invariant as a consequence of (2.16).

In appendix A we introduced the notion of an 'extensible connection'. An extensible linear connection induces a connection on the tensor product $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ which then enables us to construct 'covariant derivatives' of tensor fields. In the following we elaborate this notion for the case of linear connections $\nabla : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ on a finite group G with a bicovariant (first-order) differential calculus (with space of 1-forms Ω^1). Using proposition A.2, the following characterization of such connections is obtained.

Proposition 4.1. A linear connection on a finite group with a bicovariant (first-order) differential calculus is extensible if and only if there exist bimodule homomorphisms

$$V : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \qquad W : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \tag{4.23}$$

such that

$$\nabla\varphi = \nabla^\sigma\varphi + V(\varphi \otimes_{\mathcal{A}} \rho) + W(\varphi) \tag{4.24}$$

where ∇^σ denotes the connection (4.17) and $\rho = \theta^{\underline{g}}$.

It remains to determine the most general form of the bimodule homomorphisms V and W.

Proposition 4.2. Let (Ω^1, d) be a left-covariant first-order differential calculus on a finite group.

(a) A map $V : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is a bimodule homomorphism if and only if

$$V(\theta^g \otimes_{\mathcal{A}} \theta^{g'}) = \sum_{\substack{h,h' \in \hat{G} \\ hh' = g'g}} V^{g,g'}_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h \qquad \forall g, g' \in \hat{G} \tag{4.25}$$

with $V^{g,g'}_{h,h'} \in \mathcal{A}$.

(b) A map $W : \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ is a bimodule homomorphism if and only if

$$W(\theta^g) = \sum_{\substack{h,h' \in \hat{G} \\ hh' = g}} W^g_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h \qquad \forall g \in \hat{G} \tag{4.26}$$

with $W^g_{h,h'} \in \mathcal{A}$.

Proof. The proofs of (a) and (b) are essentially the same. We therefore only present the proof of (b). Since $\{\theta^g \otimes_{\mathcal{A}} \theta^{g'} | g, g' \in \hat{G}\}$ is a left $\mathcal{A}$-module basis of $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$, $W(\theta^g)$ must have the form

$$W(\theta^g) = \sum_{h,h' \in \hat{G}} W^g_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h$$

with coefficients in $\mathcal{A}$. This extends to a left $\mathcal{A}$-module homomorphism. The condition for W to be also right $\mathcal{A}$-linear is

$$0 = W(\theta^g) f - (R_{g^{-1}} f) W(\theta^g) = \sum_{h,h' \in \hat{G}} (R_{h'^{-1}} R_{h^{-1}} f - R_{g^{-1}} f) W^g_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h$$

for all $g \in \hat{G}$ and all $f \in \mathcal{A}$. It is then sufficient to have this property for all generators $e_{g'}$ ($g' \in G$) of $\mathcal{A}$, i.e.

$$\sum_{h,h' \in \hat{G}} (e_{g'hh'} - e_{g'g}) W^g_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h = 0.$$

This is equivalent to $W^g_{h,h'} = 0$ whenever $hh' \neq g$. □

According to the two propositions, an extensible linear connection ∇ is given by

$$\nabla \theta^g = \sum_{\substack{g',h,h' \in \hat{G} \\ hh' = g'g}} V^{g,g'}_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h + \sum_{\substack{h,h' \in \hat{G} \\ hh' = g}} W^g_{h,h'} \theta^{h'} \otimes_{\mathcal{A}} \theta^h \tag{4.27}$$

taking into account that $\nabla^\sigma \theta^g = 0$. This means

$$\Gamma^g{}_{h,h'} = \begin{cases} -W^g_{h,h'} & hh' = g \\ -V^{g,hh'g^{-1}}_{h,h'} & \text{for } g' = hh'g^{-1} \in \hat{G} \\ 0 & hh'g^{-1} \notin \hat{G} \cup \{e\} \end{cases} \tag{4.28}$$

from which we observe that there are restrictions for ∇ to be extensible iff there are products $hh'g^{-1} \notin \hat{G} \cup \{e\}$ for $h, h', g \in \hat{G}$.

Example 4. For $G = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$ and $\hat{G} = \{a, a^2\}$ we find the restrictions

$$\Gamma^{a^2}_{a,a} = \Gamma^a_{a^2,a^2} = 0 \tag{4.29}$$

for a linear connection to be extensible. This excludes the C-connection of example 3.

5. Vector fields, dual connections, and metrics on finite groups

Let Ω be a left-covariant differential calculus on a finite group G. By $\mathcal{X}$ we denote the dual $\mathcal{A}$-bimodule of Ω^1 with duality contraction $\langle \varphi, X \rangle$ for $\varphi \in \Omega^1$ and $X \in \mathcal{X}$ (see appendix B). The elements of $\mathcal{X}$ act as operators on $\mathcal{A}$ via

$$Xf := \langle \mathrm{d}f, X \rangle. \tag{5.1}$$

The (non-vanishing) Maurer–Cartan forms θ^g constitute a basis of Ω^1 as a left or right $\mathcal{A}$-module. Let $\{\ell'_g | g \in \hat{G}\}$ be the dual basis. Then

$$\ell'_g f = \langle \mathrm{d}f, \ell'_g \rangle = \langle (\ell_{\underline{h}} f) \theta^{\underline{h}}, \ell'_g \rangle = \ell_g f \tag{5.2}$$

shows that $\ell'_g = \ell_g$. In the same way one verifies that $\{r_g | g \in \hat{G}\}$ is the dual basis of $\{\omega^g\}$. Elements of $\mathcal{X}$ can now be written as

$$X = \ell_{\underline{g}} \cdot X^{\underline{g}} \tag{5.3}$$

where $Xf = (\ell_{\underline{g}} f) X^{\underline{g}}$. As a consequence of (2.12),

$$\langle \theta^h, (\mathcal{R}_g f)\ell_g - \ell_g \cdot f \rangle = \langle \theta^h (\mathcal{R}_g f), \ell_g \rangle - \delta^h_g f = (R_{h^{-1}g} f) \langle \theta^h, \ell_g \rangle - \delta^h_g f = 0 \tag{5.4}$$

so that

$$(\mathcal{R}_g f)\ell_g = \ell_g \cdot f. \tag{5.5}$$

By duality (see appendix B) a linear left $\mathcal{A}$-module connection ∇ induces a right $\mathcal{A}$-module connection ∇^* on $\mathcal{X}$ such that

$$\nabla^* \ell_g = \ell_{\underline{h}} \otimes_{\mathcal{A}} \omega^{\underline{h}}{}_g \tag{5.6}$$

where $\omega^g{}_{g'}$ are the connection 1-forms with respect to the basis θ^g (cf (4.8)). ∇^* extends to a map $\mathcal{X} \otimes_{\mathcal{A}} \Omega \to \mathcal{X} \otimes_{\mathcal{A}} \Omega$ such that

$$\nabla^*(\chi\varphi) = (\nabla^*\chi)\varphi + \chi\, \mathrm{d}\varphi \tag{5.7}$$

for $\chi \in \mathcal{X} \otimes_{\mathcal{A}} \Omega$ and $\varphi \in \Omega$. Regarding† the *canonical form*

$$\Xi := \ell_{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g}} \tag{5.8}$$

as an element of $\mathcal{X} \otimes_{\mathcal{A}} \Omega$, we find

$$\nabla^*\Xi = \nabla^*\ell_{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g}} + \ell_{\underline{g}} \otimes_{\mathcal{A}} \mathrm{d}\theta^{\underline{g}} = \ell_{\underline{g}} \otimes_{\mathcal{A}} D\theta^{\underline{g}} \tag{5.9}$$

with

$$D\theta^g := \mathrm{d}\theta^g + \omega^g{}_{\underline{h}}\theta^{\underline{h}} =: \Theta^g. \tag{5.10}$$

This is another (equivalent) expression for the torsion of ∇. Furthermore, since $\nabla^*\Xi$ is again an element of $\mathcal{X} \otimes_{\mathcal{A}} \Omega$, we can apply ∇^* another time. This yields

$$(\nabla^*)^2\Xi = \ell_{\underline{g}} \otimes_{\mathcal{A}} D\Theta^{\underline{g}} \tag{5.11}$$

where

$$D\Theta^g = \mathrm{d}\Theta^g + \omega^g{}_{\underline{g'}}\Theta^{\underline{g'}} = D^2\theta^g = \Omega^g{}_{\underline{g'}}\theta^{\underline{g'}} \tag{5.12}$$

which resembles the first Bianchi identity of classical differential geometry. We also have an analogue of the second Bianchi identity: $D\Omega^g{}_{g'} = 0$.

As a *metric* we may regard an element $\boldsymbol{g} \in \mathcal{X} \otimes_{\mathcal{A}} \mathcal{X}$ with certain properties‡. In terms of the basis $\ell_g \otimes_{\mathcal{A}} \ell_{g'}$ we have $\boldsymbol{g} = \ell_{\underline{g}} \otimes_{\mathcal{A}} \ell_{\underline{g'}} \cdot \boldsymbol{g}^{\underline{g},\underline{g'}}$. A metric is called *compatible* with a connection on $\mathcal{X} \otimes_{\mathcal{A}} \mathcal{X}$ if $\boldsymbol{g}$ is covariantly constant.

Example. For a bicovariant differential calculus, the canonical bimodule isomorphism σ has a 'dual' $\sigma' : \Omega^1 \otimes_{\mathcal{A}} \mathcal{X} \to \mathcal{X} \otimes_{\mathcal{A}} \Omega^1$ (cf (B.5)). From

$$\begin{aligned}\langle\theta^{g'}, \sigma'(\theta^h \otimes_{\mathcal{A}} \ell_g)\rangle &= \langle\sigma(\theta^{g'} \otimes_{\mathcal{A}} \theta^h), \ell_g\rangle = \langle\theta^{g'^{-1}hg'} \otimes_{\mathcal{A}} \theta^{g'}, \ell_g\rangle \\ &= \theta^{g'^{-1}hg'}\delta^{g'}_g = \langle\theta^{g'}, \ell_g \otimes_{\mathcal{A}} \theta^{g^{-1}hg}\rangle\end{aligned} \tag{5.13}$$

we deduce

$$\sigma'(\theta^h \otimes_{\mathcal{A}} \ell_g) = \ell_g \otimes_{\mathcal{A}} \theta^{g^{-1}hg}. \tag{5.14}$$

For the connection ∇^σ defined in (4.17) the dual connection (see appendix B) is given by

$$\nabla^{\sigma'}X = X \otimes_{\mathcal{A}} \rho - \sigma'(\rho \otimes_{\mathcal{A}} X) \tag{5.15}$$

and has the property $\nabla^{\sigma'}\ell_g = 0$. It is extensible and we obtain

$$\nabla^{\sigma'}_{\otimes}\boldsymbol{g} = \ell_{\underline{g}} \otimes_{\mathcal{A}} \ell_{\underline{g'}} \cdot \mathrm{d}\boldsymbol{g}^{\underline{g},\underline{g'}} \tag{5.16}$$

so that $\boldsymbol{g}$ is compatible with $\nabla^{\sigma'}_{\otimes}$ if and only if $\mathrm{d}\boldsymbol{g}^{\underline{g},\underline{g'}} = 0$§.

† We may also regard Ξ as an element of the product module $\mathcal{X} \otimes_{\mathcal{A}} \Omega^1$. A connection ∇ on Ω^1 together with its dual ∇^* can then be used to define $\tilde{\nabla}(X \otimes_{\mathcal{A}} \varphi) := (\nabla^* X) \otimes_{\mathcal{A}} \varphi + X \otimes_{\mathcal{A}} \nabla\varphi$ which implies $\tilde{\nabla}\Xi = 0$. This makes sense though $\tilde{\nabla}$ is *not* a left or right $\mathcal{A}$-module connection on $\mathcal{X} \otimes_{\mathcal{A}} \Omega^1$.

‡ A reality or hermiticity condition requires an involution, an extra structure which we leave aside in the present work. Less straightforward is the implementation of a notion of invertibility. Note that we could also think of a metric as an element of $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$. Then appendix E provides us with examples. See also the discussion in section 7.

§ The equation $\mathrm{d}f = 0$ for $f \in \mathcal{A}$ does not necessarily imply $f \in \mathbb{C}$. It depends on the differential calculus what the 'constant functions' are.

Alternatively, we can extend ∇^σ to a connection $\nabla^\sigma_\otimes$ on $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ and there is a dual $\nabla^{\sigma *}_\otimes$ of the latter (see appendix B). Then

$$\langle \theta^g \otimes_{\mathcal{A}} \theta^{g'}, \nabla^{\sigma *}_\otimes \boldsymbol{g} \rangle = \mathrm{d}\boldsymbol{g}^{g,g'} - \langle \nabla^\sigma_\otimes (\theta^g \otimes_{\mathcal{A}} \theta^{g'}), \boldsymbol{g} \rangle \tag{5.17}$$

for a metric $\boldsymbol{g}$. Since θ^g is covariantly constant with respect to ∇^σ, we again obtain $\mathrm{d}\boldsymbol{g}^{g,g'} = 0$ as the metric compatibility condition.

Let Ω^1 be the space of 1-forms of a bicovariant first-order differential calculus on a finite group. Symmetry conditions can then be imposed on a metric as follows. For example, we call $\boldsymbol{g}$ s-*symmetric* if $\langle \sigma(\varphi \otimes_{\mathcal{A}} \varphi'), \boldsymbol{g} \rangle = \langle \varphi \otimes_{\mathcal{A}} \varphi', \boldsymbol{g} \rangle$ for all $\varphi, \varphi' \in \Omega^1$. This can also be expressed as $\sigma_{\mathcal{X}} \boldsymbol{g} = \boldsymbol{g}$ in terms of the transpose $\sigma_{\mathcal{X}}$ of σ, which is determined by

$$\begin{aligned} \langle \theta^g \otimes_{\mathcal{A}} \theta^{g'}, \sigma_{\mathcal{X}}(\ell_h \otimes_{\mathcal{A}} \ell_{h'}) \rangle &= \langle \sigma(\theta^g \otimes_{\mathcal{A}} \theta^{g'}), \ell_h \otimes_{\mathcal{A}} \ell_{h'} \rangle \\ &= \langle \theta^{\mathrm{ad}(g^{-1})g'} \otimes_{\mathcal{A}} \theta^g, \ell_h \otimes_{\mathcal{A}} \ell_{h'} \rangle \\ &= \delta^g_h \delta^{\mathrm{ad}(g^{-1})g'}_{h'} = \delta^g_h \delta^{g'}_{\mathrm{ad}(h)h'} \\ &= \langle \theta^g \otimes_{\mathcal{A}} \theta^{g'}, \ell_{\mathrm{ad}(h)h'} \otimes_{\mathcal{A}} \ell_h \rangle \end{aligned} \tag{5.18}$$

i.e.

$$\sigma_{\mathcal{X}}(\ell_g \otimes_{\mathcal{A}} \ell_{g'}) = \ell_{\mathrm{ad}(g)g'} \otimes_{\mathcal{A}} \ell_g. \tag{5.19}$$

Using $r_g = \ell_{\mathrm{ad}(\underline{h}^{-1})g} \cdot e_{\underline{h}}$ and $\ell_g \cdot e_h = e_{hg^{-1}} \ell_g$ one finds $\sigma_{\mathcal{X}}(\ell_g \otimes_{\mathcal{A}} r_{g'}) = r_{g'} \otimes_{\mathcal{A}} \ell_g$ which is the analogue of (3.1). Indeed, $\mathcal{X}$ inherits from Ω^1 the structure of a bicovariant bimodule and $\sigma_{\mathcal{X}}$ is the corresponding braid operator.

According to the general construction in appendix C, a left (right) coaction on Ω^1 induces a left (right) coaction on $\mathcal{X}$. For a left-covariant first-order differential calculus the left coaction on $\mathcal{X}$ is determined by

$$\Delta_{\mathcal{X}}(\ell_g) = \mathbb{1} \otimes \ell_g \tag{5.20}$$

i.e. the elements dual to the left-invariant basis 1-forms θ^g are also left-invariant. An element $X = \ell_{\underline{g}} \cdot X^{\underline{g}}$ of $\mathcal{X}$ is left-invariant iff $X^g \in \mathbb{C}$ for all $g \in \hat{G}$. The coaction extends to $\mathcal{X} \otimes_{\mathcal{A}} \mathcal{X}$ (see appendix C). A tensor $\boldsymbol{g} = \ell_{\underline{g}} \otimes_{\mathcal{A}} \ell_{\underline{g}'} \cdot \boldsymbol{g}^{\underline{g},\underline{g}'}$ is then left-invariant iff $\boldsymbol{g}^{\underline{g},\underline{g}'} \in \mathbb{C}$. The above example now shows that every left-invariant metric is covariantly constant with respect to (the extension of) the connection $\nabla^{\sigma'}$.

6. Non-commutative geometry of the symmetric group $\mathcal{S}_3$

We denote the elements of the symmetric group $\mathcal{S}_3$ as follows,

$$a = (12) \qquad b = (23) \qquad c = (13) \tag{6.1}$$

ab, ba, and e (the unit element). In order to determine the $\mathcal{S}_3$ left-invariant differential calculi on a set of six elements, we have to calculate the orbits of the left-action on $(\mathcal{S}_3 \times \mathcal{S}_3)'$. These are

$$\begin{aligned} \mathcal{O}_1 &= \{(e, a), (a, e), (b, ba), (c, ab), (ab, c), (ba, b)\} \\ \mathcal{O}_2 &= \{(e, b), (a, ab), (b, e), (c, ba), (ab, a), (ba, c)\} \\ \mathcal{O}_3 &= \{(e, c), (a, ba), (b, ab), (c, e), (ab, b), (ba, a)\} \\ \mathcal{O}_4 &= \{(e, ab), (a, b), (b, c), (c, a), (ab, ba), (ba, e)\} \\ \mathcal{O}_5 &= \{(e, ba), (a, c), (b, a), (c, b), (ab, e), (ba, ab)\}. \end{aligned}$$

They are in correspondence with elements of $G \setminus \{e\}$. Left-invariant differential calculi are obtained by deleting subgraphs corresponding to orbits from the graph corresponding to the universal differential calculus (which is left-invariant, of course).

With respect to left- and right-action $(\mathcal{S}_3 \times \mathcal{S}_3)'$ decomposes into two orbits†,

$$\mathcal{O}_{\mathrm{I}} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \qquad \mathcal{O}_{\mathrm{II}} = \mathcal{O}_4 \cup \mathcal{O}_5. \tag{6.2}$$

Hence, there are two bicovariant differential calculi, Ω^1_{I} and Ω^1_{II}, on $\mathcal{S}_3$ besides the universal and the trivial one. Their graphs are obtained by deleting all arrows corresponding to elements of $\mathcal{O}_{\mathrm{I}}$ or $\mathcal{O}_{\mathrm{II}}$, respectively, from the graph associated with the universal differential calculus (see figure 2). All these graphs are symmetric in the sense that for every arrow also the reverse arrow is present.

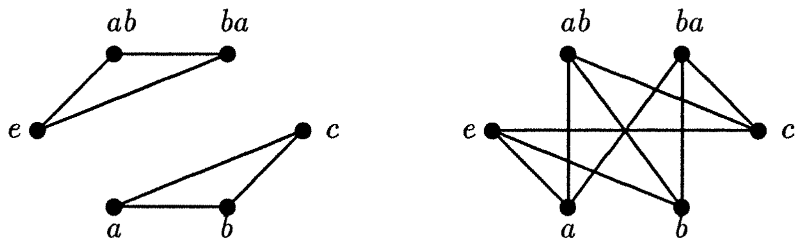

Figure 2. The graphs which determine the bicovariant first-order differential calculi Ω^1_{I} and Ω^1_{II} on $\mathcal{S}_3$.

The bimodule isomorphism σ is a non-trivial map in the case under consideration. In terms of the decomposition into conjugacy classes

$$\mathcal{S}_3 = \{e\} \cup \underbrace{\{a, b, c\}}_{=:\mathcal{S}_3'} \cup \underbrace{\{ab, ba\}}_{=:\mathcal{S}_3''} \tag{6.3}$$

it is given by

$$\begin{aligned}
&\sigma(\theta^x, \theta^x) = \theta^x \otimes_{\mathcal{A}} \theta^x && \forall x \in \mathcal{S}_3 \\
&\sigma(\theta^x, \theta^y) = \theta^z \otimes_{\mathcal{A}} \theta^x && \forall x, y \in \mathcal{S}_3', x \neq y, z \in \mathcal{S}_3' \setminus \{x, y\} \\
&\sigma(\theta^{ab}, \theta^{ba}) = \theta^{ba} \otimes_{\mathcal{A}} \theta^{ab} && \\
&\sigma(\theta^{ba}, \theta^{ab}) = \theta^{ab} \otimes_{\mathcal{A}} \theta^{ba} && \\
&\sigma(\theta^x, \theta^{ab}) = \theta^{ba} \otimes_{\mathcal{A}} \theta^x && \forall x \in \mathcal{S}_3' \\
&\sigma(\theta^x, \theta^{ba}) = \theta^{ab} \otimes_{\mathcal{A}} \theta^x && \forall x \in \mathcal{S}_3' \\
&\sigma(\theta^{ab}, \theta^x) = \theta^y \otimes_{\mathcal{A}} \theta^{ab} && \text{for } (x, y) \in \{(a, c), (b, a), (c, b)\} \\
&\sigma(\theta^{ba}, \theta^x) = \theta^y \otimes_{\mathcal{A}} \theta^{ba} && \text{for } (x, y) \in \{(a, b), (b, c)(c, a)\}
\end{aligned} \tag{6.4}$$

for the universal first-order differential calculus. Since the centre of $\mathcal{S}_3$ is trivial, $\mathrm{ad}(\mathcal{S}_3) \cong \mathcal{S}_3$ and $|\,\mathrm{ad}(\mathcal{S}_3)| = 6$, so that $\sigma^{12} = \mathrm{id}$ according to (3.6). For the other bicovariant calculi, the corresponding σ is induced in an obvious way. In the case of Ω^1_{I} the bimodule isomorphism is the one of the commutative (sub)group $\mathbb{Z}_3$. Hence, $\sigma_{\mathrm{I}}^2 = \mathrm{id}$. For Ω^1_{II} one can deduce from (6.4) that $\sigma_{\mathrm{II}}^3 = \mathrm{id}$. For the restriction of σ to the sub-bimodule of $\tilde{\Omega}^1 \otimes_{\mathcal{A}} \tilde{\Omega}^1$ which is generated by $\{\theta^x \otimes_{\mathcal{A}} \theta^{x'}, \theta^{x'} \otimes_{\mathcal{A}} \theta^x | x \in \mathcal{S}_3', x' \in \mathcal{S}_3''\}$ we have $\sigma^4 = \mathrm{id}$. Hence $2|\,\mathrm{ad}(\mathcal{S}_3)| = 12$ is actually the order of σ for the universal first-order differential calculus, in accordance with proposition 3.2.

† These orbits are in correspondence with the non-trivial conjugacy classes in $\mathcal{S}_3$, i.e. $\{a, b, c\}$ and $\{ab, ba\}$ (cf section 2).

In order to determine the most general bi-invariant linear connection on $\mathcal{S}_3$ with the universal first-order differential calculus, one has to determine the $(\mathrm{ad})^3$-orbits in $(G \setminus \{e\})^3$. There are 24 of them so that bi-invariance restricts the 125 connection coefficients in (4.7) to 24 independent constants.

6.1. Geometry of the three-dimensional bicovariant calculus

The bimodule Ω^1_{II} is generated (as a left $\mathcal{A}$-module) by $\theta^a, \theta^b, \theta^c$. The operator σ acts on $\Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}}$ as follows,

$$\begin{aligned} &\sigma(\theta^x \otimes_{\mathcal{A}} \theta^x) = \theta^x \otimes_{\mathcal{A}} \theta^x \qquad \forall x \in \{a, b, c\} \\ &\sigma(\theta^x \otimes_{\mathcal{A}} \theta^y) = \theta^z \otimes_{\mathcal{A}} \theta^x \end{aligned} \tag{6.5}$$

where, in the last equation, z is the complement of x, y in $\mathcal{S}'_3$. As already mentioned, $\sigma^2 \neq \mathrm{id}$, but $\sigma^3 = \mathrm{id}$. This rules out -1 as an eigenvalue of σ so that all elements of $\Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}}$ are w-symmetric according to proposition 3.3. The bimodule $\Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}}$ splits into a direct sum of sub-bimodules (cf proposition 3.3), $\Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}} = \ker \boldsymbol{A} \oplus \mathrm{im}\, \boldsymbol{A}$ where $\ker \boldsymbol{A}$ is generated by $\theta^x \otimes_{\mathcal{A}} \theta^x$ for $x \in \{a, b, c\}$, $\theta^a \otimes_{\mathcal{A}} \theta^b + \theta^b \otimes_{\mathcal{A}} \theta^c + \theta^c \otimes_{\mathcal{A}} \theta^a$ and $\theta^b \otimes_{\mathcal{A}} \theta^a + \theta^a \otimes_{\mathcal{A}} \theta^c + \theta^c \otimes_{\mathcal{A}} \theta^b$. These are eigenvectors of σ with eigenvalue one, i.e. s-symmetric tensors. The image of $\Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}}$ under $\boldsymbol{A}$ is generated by $\theta^a \otimes_{\mathcal{A}} \theta^b - \theta^c \otimes_{\mathcal{A}} \theta^a$, $\theta^a \otimes_{\mathcal{A}} \theta^b - \theta^b \otimes_{\mathcal{A}} \theta^c$, $\theta^a \otimes_{\mathcal{A}} \theta^c - \theta^b \otimes_{\mathcal{A}} \theta^a$ and $\theta^a \otimes_{\mathcal{A}} \theta^c - \theta^c \otimes_{\mathcal{A}} \theta^b$. These are w-antisymmetric tensors. The space of 2-forms (following Woronowicz) is therefore four-dimensional. A basis is given by $\theta^a\theta^b, \theta^b\theta^c, \theta^a\theta^c, \theta^b\theta^a$ and we have the relations

$$\theta^c\theta^a = -\theta^a\theta^b - \theta^b\theta^c \qquad \theta^c\theta^b = -\theta^b\theta^a - \theta^a\theta^c \qquad \theta^x\theta^x = 0 \qquad x \in \{a, b, c\}. \tag{6.6}$$

For a linear connection, there are *a priori* 27 connection coefficients. Bi-invariance restricts them as follows:

$$\begin{aligned} &\Gamma^a_{a,a} = \Gamma^b_{b,b} = \Gamma^c_{c,c} \\ &\Gamma^a_{a,b} = \Gamma^a_{a,c} = \Gamma^b_{b,a} = \Gamma^b_{b,c} = \Gamma^c_{c,a} = \Gamma^c_{c,b} \\ &\Gamma^a_{b,a} = \Gamma^a_{c,a} = \Gamma^b_{a,b} = \Gamma^b_{c,b} = \Gamma^c_{a,c} = \Gamma^c_{b,c} \\ &\Gamma^a_{b,b} = \Gamma^a_{c,c} = \Gamma^b_{a,a} = \Gamma^b_{c,c} = \Gamma^c_{a,a} = \Gamma^c_{b,b} \\ &\Gamma^a_{b,c} = \Gamma^a_{c,b} = \Gamma^b_{a,c} = \Gamma^b_{c,a} = \Gamma^c_{a,b} = \Gamma^c_{b,a}. \end{aligned} \tag{6.7}$$

The condition of vanishing torsion for a bi-invariant connection becomes

$$\Gamma^a_{b,a} = \Gamma^a_{a,b} = \Gamma^a_{b,c} - 1 \tag{6.8}$$

leaving us with only three independent constants. It turns out that ${\Gamma^x}_{y,z} = {\Gamma^x}_{z,y}$ for bi-invariant connections without torsion. Among these is the C-connection (4.22) for which $\Gamma^a_{a,a} = -2$, $\Gamma^a_{a,b} = -1$, $\Gamma^a_{b,b} = \Gamma^a_{b,c} = 0$. There are no bi-invariant connections for which torsion and curvature vanish.

In the case under consideration, we have $\hat{G} = \{a, b, c\} = \hat{G}^{-1}$, $\hat{G} \cdot \hat{G} = \{e, ab, ba\}$ and $\hat{G} \cdot \hat{G} \cdot \hat{G} = \hat{G}$. As a consequence, every linear connection is extensible (cf section 4). But there are no (non-trivial) bimodule homomorphisms $W : \Omega^1_{\mathrm{II}} \to \Omega^1_{\mathrm{II}} \otimes_{\mathcal{A}} \Omega^1_{\mathrm{II}}$ since $gg' \notin \hat{G}$ for all $g, g' \in \hat{G}$. From proposition 4.1 we infer that all linear connections have the form

$$\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi + V(\varphi \otimes_{\mathcal{A}} \rho) \tag{6.9}$$

with a bimodule homomorphism V. This includes the family (4.18), of course, which in the case under consideration depends on three independent constants. The only linear (left $\mathcal{A}$-module) connection with right $\mathcal{A}$-linearity is given by $\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi$ in accordance with proposition A.3.

7. Final remarks

In this work we have continued our previous research on non-commutative geometry of discrete sets [4, 5] and, in particular, finite groups [8, 9].

Much of the material presented concerns the notion of linear connections. In [18] a special class of linear left $\mathcal{A}$-module connections has been considered satisfying the additional condition

$$\nabla(\varphi f) = \tau(\varphi \otimes_{\mathcal{A}} \mathrm{d} f) + (\nabla\varphi) f \tag{7.1}$$

where τ is an $\mathcal{A}$-bimodule homomorphism. In classical differential geometry, all linear connections satisfy this condition with the choice of the permutation map for τ. This observation was taken in [18] to consider the above condition in non-commutative geometry. It should be noticed, however, that connections in commutative geometry *automatically* satisfy this condition whereas in non-commutative geometry it severely restricts the possible (linear) connections, in general. It is therefore quite unclear from this point of view what the relevance of the class of (linear) connections determined by (7.1) is. However, it has also been pointed out in [18] that linear connections with the above property can be extended to tensor products (over $\mathcal{A}$) of 1-forms. Indeed, given connections on two $\mathcal{A}$-bimodules, it seems to be impossible, in general, to build from these a connection on the tensor product of the two modules. In particular, we would like to achieve this in order to be able to talk about a covariantly constant metric. In appendix A we have addressed the question of extensibility in more generality. In our attempt to solve this problem, we were led to the condition (7.1), which we therefore called 'extensibility condition'. This provides a much stronger motivation for the consideration of the special class of linear connections satisfying (7.1). We have to stress, however, that there may still be a way beyond our ansatz to extend connections. In appendix D we briefly discussed a natural modification of the usual definition of a linear connection which guarantees extensibility. It turned out to be too restrictive, however.

For a bicovariant differential calculus on a Hopf algebra there is a canonical choice for τ, the canonical bimodule isomorphism σ [7]. Using the fact that powers of σ are again bimodule isomorphisms, one actually has a whole class of extensible linear connections on the Hopf algebra. Similar observations have been made in [16] where, however, the restriction to 'generalized permutations' τ which satisfy $\pi \circ (\tau + \mathrm{id}) = 0$ rules out $\tau = \sigma$ (together with $\pi = (\mathrm{id} - \sigma)/2$ which is used in [7] to extend bicovariant first-order differential calculi on Hopf algebras to higher orders) if $\sigma^2 \neq \mathrm{id}$. The reasoning behind this restriction (see also [19]) is not quite transparent for us and in the formalism presented in this paper (which extends beyond finite groups) there is no natural place for it.

We should stress that extensibility conditions for connections not only arise for non-commutative algebras, but already for commutative algebras with 'non-commutative differential calculi' (where functions do not commute with 1-forms, in general), hence in particular for differential calculi on finite sets. For finite groups we have elaborated the extension condition for linear connections and worked out the corresponding restrictions.

The non-commutativity of a differential calculus with space of 1-forms Ω^1 results in a non-locality of the tensor product of 1-forms. This manifests itself in the fact that components of an element $\alpha \in \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ with respect to some (left or right $\mathcal{A}$-module) basis of Ω^1 do not transform in a covariant manner under a change of basis†. Though from a mathematical point of view one can hardly think of an alternative of the tensor

† This is in contrast to the fact that other basic constructions in non-commutative geometry indeed lead to quantities with covariant components, see section 4.

product over the algebra $\mathcal{A}$, it may not be (directly) suitable for a description of physics. It seems that some modification is needed. An example is provided by [20] where, for a certain non-commutative differential calculus on manifolds, a modified wedge product was constructed with the help of a linear connection, which then allowed one to read off covariant components from (generalized) differential forms.

It is the problem just mentioned which indicates that in non-commutative geometry the concept of a metric as an element of $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$, respectively a dual module, may be too naive. A departure from this concept might have crucial consequences for the relevance of the class of extensible linear connections, of course. Further exploration of (finite) examples, and perhaps even those presented in this work, should shed more light on these problems.

As we have seen in section 4, there are examples in the class of extensible connections which have a geometrical meaning as analogues of the $(\pm)$-parallelisms on Lie groups. More generally, corresponding connections exist on every Hopf algebra with a bicovariant differential calculus which is inner with a bi-invariant 1-form ρ, so in particular on the quantum groups $GL_q(n)$ (see also [16]). On the other hand, naturally associated with a left-covariant differential calculus on a finite group is the C-connection (introduced in section 4) which is not extensible, in general.

We have developed 'differential geometry' on finite sets to a level which now enables us to write down 'geometric equations' on discrete differential manifolds and to look for exact solutions. Comparatively simple examples are given by the equations of vanishing curvature or vanishing torsion for a linear connection, in which cases we presented exact solutions. More interesting would be an analogue of Einstein's equations, of course, but, as mentioned above, there is still something to be understood concerning the concept of a metric before we can seriously proceed towards this goal.

Acknowledgment

FM-H is grateful to Professor H L deVries for a helpful discussion on symmetric groups.

Appendix A. Extension of connections to tensor products of bimodules

Let $\mathcal{A}$ be an associative unital algebra, Γ an $\mathcal{A}$-bimodule, Γ' a left $\mathcal{A}$-module, and ∇, ∇' left $\mathcal{A}$-module connections on Γ and Γ', respectively, with respect to a first-order differential calculus on $\mathcal{A}$ with space of 1-forms Ω^1. We would like to build from these a connection on $\Gamma \otimes_{\mathcal{A}} \Gamma'$, i.e. a map

$$\nabla_\otimes : \Gamma \otimes_{\mathcal{A}} \Gamma' \to \Omega^1 \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Gamma' \tag{A.1}$$

which is $\mathbb{C}$-linear and satisfies

$$\nabla_\otimes(f(\gamma \otimes_{\mathcal{A}} \gamma')) = \mathrm{d}f \otimes_{\mathcal{A}} \gamma \otimes_{\mathcal{A}} \gamma' + f\nabla_\otimes(\gamma \otimes_{\mathcal{A}} \gamma') \qquad \forall f \in \mathcal{A}, \gamma \in \Gamma, \gamma' \in \Gamma'. \tag{A.2}$$

In order to be well defined, the extended connection has to satisfy

$$\nabla_\otimes(\gamma f \otimes_{\mathcal{A}} \gamma') = \nabla_\otimes(\gamma \otimes_{\mathcal{A}} f\gamma'). \tag{A.3}$$

Let us consider the ansatz

$$\nabla_\otimes = \Phi \circ (\nabla \otimes \mathrm{id}_{\Gamma'}) + \Psi \circ (\mathrm{id}_\Gamma \otimes \nabla') \tag{A.4}$$

with linear maps

$$\Phi : \Omega^1 \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Gamma' \to \Omega^1 \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Gamma'$$
$$\Psi : \Gamma \otimes_{\mathcal{A}} \Omega^1 \otimes_{\mathcal{A}} \Gamma' \to \Omega^1 \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Gamma'.$$

In the following we evaluate the conditions which the extended connection has to satisfy. These restrict the possibilities for the maps Φ and Ψ. From (A.3) we obtain the following condition:

$$\Psi(\gamma \otimes_{\mathcal{A}} \mathrm{d}f \otimes_{\mathcal{A}} \gamma') = \Phi([\nabla(\gamma f) - (\nabla\gamma) f] \otimes_{\mathcal{A}} \gamma'). \tag{A.5}$$

(A.2) leads to

$$\begin{aligned} 0 = \Phi(\mathrm{d}f \otimes_{\mathcal{A}} \gamma \otimes_{\mathcal{A}} \gamma') - \mathrm{d}f \otimes_{\mathcal{A}} \gamma \otimes_{\mathcal{A}} \gamma' + \Phi(f \nabla\gamma \otimes_{\mathcal{A}} \gamma') \\ - f\Phi(\nabla\gamma \otimes_{\mathcal{A}} \gamma') + \Psi(f\gamma \otimes_{\mathcal{A}} \nabla'\gamma') - f\Psi(\gamma \otimes_{\mathcal{A}} \nabla'\gamma'). \end{aligned} \tag{A.6}$$

If we demand Φ and Ψ to be left $\mathcal{A}$-linear, then the last equation implies

$$\Phi = \mathrm{id}_{\Omega^1} \otimes \mathrm{id}_{\Gamma} \otimes \mathrm{id}_{\Gamma'} \tag{A.7}$$

and (A.5) reduces to

$$\Psi = \Psi_{\nabla} \otimes \mathrm{id}_{\Gamma'} \tag{A.8}$$

with a map $\Psi_{\nabla} : \Gamma \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ such that

$$\Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} \mathrm{d}f) = \nabla(\gamma f) - (\nabla\gamma) f \qquad \forall f \in \mathcal{A}, \gamma \in \Gamma. \tag{A.9}$$

If we could turn this into a definition, we would have a universal solution to the problem we started with, the extension of connections on two bimodules to the tensor product (over $\mathcal{A}$)†. As a consequence of our assumptions, Ψ_{∇} is defined on $\Gamma \otimes_{\mathcal{A}} \Omega^1$, but the right-hand side of (A.9) must not respect that. This means that, depending on the chosen differential calculus on $\mathcal{A}$, (A.9) is only well defined for a special class of connections on Γ.

Definition. A connection $\nabla : \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is called *extensible*‡ if it defines a map Ψ_{∇} via (A.9).

Restrictions arise as follows. A relation $\sum h_k \,\mathrm{d}f_k = 0$ in Ω^1 implies $\sum [\nabla(\gamma h_k f_k) - \nabla(\gamma h_k) f_k] = 0$. If Ω^1 is the space of 1-forms of the universal first-order differential calculus on $\mathcal{A}$, there are no relations of the form $\sum h_k \mathrm{d} f_k = 0$ and therefore every connection is extensible.

Lemma A.1. For an extensible connection, Ψ_{∇} is an $\mathcal{A}$-bimodule homomorphism.

Proof.

$$\begin{aligned} \Psi_{\nabla}(h\gamma \otimes_{\mathcal{A}} \mathrm{d}f) &= \nabla(h\gamma f) - \nabla(h\gamma) f \\ &= \mathrm{d}h \otimes_{\mathcal{A}} (\gamma f) + h\nabla(\gamma f) - (\mathrm{d}h \otimes_{\mathcal{A}} \gamma) f - (h\nabla\gamma) f \\ &= h[\nabla(\gamma f) - (\nabla\gamma) f] \\ &= h\Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} \mathrm{d}f) \\ \Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} (\mathrm{d}f)h) &= \Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} \mathrm{d}(fh)) - \Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} f\mathrm{d}h) \\ &= [\nabla(\gamma f) - (\nabla\gamma) f]h \\ &= \Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} \mathrm{d}f)h \end{aligned}$$

for all $f, h \in \mathcal{A}$. □

What we have shown so far is summarized in the following proposition.

† Here and in the following we shall assume that the tensor product over $\mathcal{A}$ is zero divisor free. (As an example, the tensor product over $\mathbb{Z}$ of elements of $\mathbb{Z}_n$ with rational numbers always vanishes). If we do not make this assumption, there are additional consistency conditions for the connection $\nabla_{\otimes}$. Let $\Gamma_0 := \{\gamma \in \Gamma | \gamma \otimes_{\mathcal{A}} \gamma' = 0 \forall \gamma' \in \Gamma'\}$ and Γ'_0 defined correspondingly. Then we have to ensure that $\nabla\Gamma_0 \subset \Omega^1 \otimes_{\mathcal{A}} \Gamma_0$ and $\nabla'\Gamma'_0 \subset \Omega^1 \otimes_{\mathcal{A}} \Gamma'_0$.

‡ In the notation of [21] an extensible connection is a 'bimodule connection'.

Proposition A.1. For left $\mathcal{A}$-module connections ∇, ∇' on $\mathcal{A}$-bimodules Γ, Γ' (with respect to a first-order differential calculus on $\mathcal{A}$) there exists a connection on $\Gamma \otimes_{\mathcal{A}} \Gamma'$ of the form

$$\nabla_{\otimes} = \Phi \circ (\nabla \otimes \mathrm{id}_{\Gamma'}) + \Psi \circ (\mathrm{id}_{\Gamma} \otimes \nabla') \tag{A.10}$$

with left $\mathcal{A}$-module homomorphisms Φ and Ψ if and only if ∇ is extensible. The connection is then unique and given by

$$\nabla_{\otimes} = \nabla \otimes \mathrm{id}_{\Gamma'} + (\Psi_{\nabla} \otimes \mathrm{id}_{\Gamma'}) \circ (\mathrm{id}_{\Gamma} \otimes \nabla') \tag{A.11}$$

with the $\mathcal{A}$-bimodule homomorphism Ψ_{∇} defined via (A.9).

If $\nabla : \Gamma \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is an extensible connection on an $\mathcal{A}$-bimodule Γ, a connection on the n-fold tensor product Γ^n of Γ (over $\mathcal{A}$) is inductively defined via

$$\nabla_{\otimes^n} := \nabla \otimes \mathrm{id}_{\Gamma^{n-1}} + (\Psi_{\nabla} \otimes \mathrm{id}_{\Gamma^{n-1}}) \circ (\mathrm{id}_{\Gamma} \otimes \nabla^{(n-1)}). \tag{A.12}$$

For an extensible connection we simply regard (A.9) as the definition of Ψ_{∇}. If, however, we choose some bimodule homomorphism for Ψ_{∇} on the left-hand side of (A.9), then this imposes further constraints on the connection. Corresponding examples appeared in [18, 19].

Proposition A.2. Let $\mathcal{A}$ be an associative algebra and (Ω^1, d) a first-order differential calculus on $\mathcal{A}$ which is inner, i.e. there is a 1-form ρ such that $\mathrm{d}f = [\rho, f]$ $(\forall f \in \mathcal{A})$. Let $\sigma : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ be a bimodule homomorphism†.

A linear connection is then extensible if and only if there exist bimodule homomorphisms

$$V : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \qquad W : \Omega^1 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \tag{A.13}$$

such that

$$\nabla\varphi = \nabla^{\sigma}\varphi + V(\varphi \otimes_{\mathcal{A}} \rho) + W(\varphi) \tag{A.14}$$

where ∇^{σ} denotes the linear connection associated‡ with σ, defined by $\nabla^{\sigma}\varphi := \rho \otimes_{\mathcal{A}} \varphi - \sigma(\varphi \otimes_{\mathcal{A}} \rho)$.

Proof. '$\Rightarrow$'. For an extensible ∇ there is a bimodule homomorphism $\Psi_{\nabla} : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ such that $\nabla(\varphi f) = (\nabla\varphi) f + \Psi_{\nabla}(\varphi \otimes_{\mathcal{A}} \mathrm{d}f)$. Then

$$\nabla^{\Psi_{\nabla}}\varphi := \rho \otimes_{\mathcal{A}} \varphi - \Psi_{\nabla}(\varphi \otimes_{\mathcal{A}} \rho)$$

defines a linear connection and the difference $W := \nabla - \nabla^{\Psi_{\nabla}}$ is a bimodule homomorphism. With the bimodule homomorphism $V := \sigma - \Psi_{\nabla}$ we obtain the decomposition (A.14).

'$\Leftarrow$'. Assuming that (A.14) holds, we get

$$\begin{aligned}\nabla(\varphi f) - (\nabla\varphi) f &= \nabla^{\sigma}(\varphi f) - (\nabla^{\sigma}\varphi) f + V(\varphi f \otimes_{\mathcal{A}} \rho) - V(\varphi \otimes_{\mathcal{A}} \rho) f \\ &= (\sigma - V)(\varphi \otimes_{\mathcal{A}} \mathrm{d}f).\end{aligned}$$

Since $\Psi_{\nabla} := \sigma - V$ is a bimodule homomorphism, ∇ is extensible. □

We should stress the following. The notion of extensibility of a connection is based on the ansatz (A.4). We cannot exclude yet that there is a (more complicated) recipe to extend connections to the tensor product of the modules on which they live, without imposing restrictions on the connections. We have tried out several modifications of (A.4) without success.

† A possible choice is $\sigma \equiv 0$.
‡ See also [16].

In [14] a class of left $\mathcal{A}$-module connections on a bimodule has been considered with additional right $\mathcal{A}$-linearity†. This is a subclass of extensible connections. For an algebra $\mathcal{A}$ with an inner first-order differential calculus, there always exists one particular connection of this kind, the *canonical (left $\mathcal{A}$-module) connection* which is given by $\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi$. A complete characterization of such connections is obtained in the following proposition.

Proposition A.3. Let $\mathcal{A}$ be an associative algebra and (Ω^1, d) a first-order differential calculus on $\mathcal{A}$ which is inner (with a 1-form ρ). Then every left $\mathcal{A}$-module connection which is also a right $\mathcal{A}$-module homomorphism has the form

$$\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi + W(\varphi) \qquad \forall\varphi \in \Omega^1 \tag{A.15}$$

with a bimodule homomorphism W.

Proof. A left $\mathcal{A}$-module connection with the right $\mathcal{A}$-module homomorphism property $\nabla(\varphi f) = (\nabla\varphi) f$ is a special case of an extensible connection (with $\Psi_\nabla = 0$). Proposition A.2 then tells us that $\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi + V(\varphi \otimes_{\mathcal{A}} \rho) + W(\varphi)$ with bimodule homomorphisms V and W. The difference of two left $\mathcal{A}$-module connections with right $\mathcal{A}$-linearity must be an $\mathcal{A}$-bimodule homomorphism $\Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$. A simple calculation using $\mathrm{d}f = [\rho, f]$ then shows that V has to vanish. □

The constraint imposed by (A.15) on a connection is very restrictive. In section 6.1 we have an example where the canonical left $\mathcal{A}$-module connection turns out to be the only left $\mathcal{A}$-module connection with right $\mathcal{A}$-linearity. Further examples are provided by bicovariant first-order differential calculi on the quantum groups $GL_q(n)$‡. In this case it has been shown [16] that there is no non-vanishing bimodule homomorphism $\Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$. All these calculi are inner [7] (with a 1-form ρ). Hence, the canonical connection $\nabla\varphi = \rho \otimes_{\mathcal{A}} \varphi$ is the only left $\mathcal{A}$-module connection with right $\mathcal{A}$-linearity according to proposition A.3.

Appendix B. Connections and their duals

Let $\mathcal{A}$ be an associative algebra and Γ an $\mathcal{A}$-bimodule. There are two natural ways to define a *dual* of Γ, depending on whether its elements act from the left or from the right on elements of Γ. Here we make the latter choice (see the remark at the end of this section). For the duality contraction $\langle\gamma, \mu\rangle$ where $\gamma \in \Gamma$ and $\mu \in \Gamma^*$ (the dual of Γ), we then have

$$\langle f\gamma, \mu\rangle = f\langle\gamma, \mu\rangle \qquad \langle\gamma, \mu f\rangle = \langle\gamma, \mu\rangle f \qquad \langle\gamma f, \mu\rangle = \langle\gamma, f\mu\rangle \tag{B.1}$$

for all $f \in \mathcal{A}$. For a left $\mathcal{A}$-module connection on Γ (with respect to some first-order differential calculus on $\mathcal{A}$ with space of 1-forms Ω^1) its dual is a map $\nabla^* : \Gamma^* \to \Gamma^* \otimes_{\mathcal{A}} \Omega^1$ defined by

$$\langle\gamma, \nabla^*\mu\rangle := \mathrm{d}\langle\gamma, \mu\rangle - \langle\nabla\gamma, \mu\rangle \tag{B.2}$$

where $\langle\gamma, \mu \otimes_{\mathcal{A}} \varphi\rangle := \langle\gamma, \mu\rangle\varphi$ and $\langle\varphi \otimes_{\mathcal{A}} \gamma, \mu\rangle := \varphi\langle\gamma, \mu\rangle$. With these definitions we obtain

$$\begin{aligned}\langle\gamma, \nabla^*(\mu f)\rangle &= \mathrm{d}\langle\gamma, \mu f\rangle - \langle\nabla\gamma, \mu f\rangle \\ &= (\mathrm{d}\langle\gamma, \mu\rangle) f + \langle\gamma, \mu\rangle \mathrm{d}f - \langle\nabla\gamma, \mu\rangle f \\ &= \langle\gamma, \nabla^*\mu\rangle f + \langle\gamma, \mu \otimes_{\mathcal{A}} \mathrm{d}f\rangle\end{aligned} \tag{B.3}$$

† The authors of [14] call such connections *left connections*. Furthermore, a 'connection on a bimodule' is defined in [14] as a pair of left and right connections.
‡ A classification of bicovariant differential calculi on the quantum general linear groups $GL_q(n)$ has been obtained in [22].

and therefore

$$\nabla^*(\mu f) = (\nabla^*\mu) f + \mu \otimes_{\mathcal{A}} \mathrm{d}f \tag{B.4}$$

which shows that ∇^* is a right $\mathcal{A}$-module connection. For an extensible connection (see appendix A) we now have the following result.

Proposition B.1. If $\nabla : \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is an extensible connection with bimodule homomorphism Ψ_∇, then ∇^* is an extensible connection with bimodule homomorphism $\Psi_{\nabla^*} : \Omega^1 \otimes_{\mathcal{A}} \Gamma^* \to \Gamma^* \otimes_{\mathcal{A}} \Omega^1$ given by

$$\langle \gamma, \Psi_{\nabla^*}(\varphi \otimes_{\mathcal{A}} \mu)\rangle := \langle \Psi_\nabla(\gamma \otimes_{\mathcal{A}} \varphi), \mu\rangle. \tag{B.5}$$

Proof. It is easily checked that Ψ_{∇^*} is well defined via (B.5) and that it is a bimodule homomorphism. We still have to verify that Ψ_{∇^*} satisfies the counterpart of (A.9) for a right $\mathcal{A}$-module connection,

$$\begin{aligned}\langle \gamma, \nabla^*(f\mu) - f\nabla^*\mu\rangle &= \langle \gamma, \nabla^*(f\mu)\rangle - \langle \gamma f, \nabla^*\mu\rangle \\ &= \mathrm{d}\langle \gamma, f\mu\rangle - \langle \nabla\gamma, f\mu\rangle - \mathrm{d}\langle \gamma f, \mu\rangle + \langle \nabla(\gamma f), \mu\rangle \\ &= \langle \nabla(\gamma f) - (\nabla\gamma) f, \mu\rangle = \langle \Psi_\nabla(\gamma \otimes_{\mathcal{A}} \mathrm{d}f), \mu\rangle.\end{aligned}$$

□

Let Γ' be a left $\mathcal{A}$-module and $\nabla' : \Gamma' \to \Omega^1 \otimes_{\mathcal{A}} \Gamma'$ a connection on it. Its dual Γ'^* is a right $\mathcal{A}$-module and $\nabla'^* : \Gamma'^* \to \Gamma'^* \otimes_{\mathcal{A}} \Omega^1$ defined as above is a connection on Γ'^*. In case we have on Γ an extensible left $\mathcal{A}$-module connection with a bimodule homomorphism Ψ_∇, we can define a connection $\nabla_\otimes$ on $\Gamma \otimes_{\mathcal{A}} \Gamma'$ in terms of the connections on Γ and Γ' (see appendix A). In the following we have to assume that the dual module of $\Gamma \otimes_{\mathcal{A}} \Gamma'$ is isomorphic to $\Gamma'^* \otimes_{\mathcal{A}} \Gamma^*$. This holds in particular for modules of finite rank. The duality contraction is then given by

$$\langle \gamma \otimes_{\mathcal{A}} \gamma', \nu \otimes_{\mathcal{A}} \mu\rangle := \langle \gamma\langle \gamma', \nu\rangle, \mu\rangle. \tag{B.6}$$

Now we have two different ways to define a connection on $\Gamma'^* \otimes_{\mathcal{A}} \Gamma^*$, either as the dual of $\nabla_\otimes$, i.e.

$$\langle \gamma \otimes_{\mathcal{A}} \gamma', (\nabla_\otimes)^*(\nu \otimes_{\mathcal{A}} \mu)\rangle := \mathrm{d}\langle \gamma \otimes_{\mathcal{A}} \gamma', \nu \otimes_{\mathcal{A}} \mu\rangle - \langle \nabla_\otimes(\gamma \otimes_{\mathcal{A}} \gamma'), \nu \otimes_{\mathcal{A}} \mu\rangle \tag{B.7}$$

or as the 'tensor product' of the duals ∇^* and ∇'^*, i.e.

$$(\nabla^*)_\otimes(\nu \otimes_{\mathcal{A}} \mu) := (\mathrm{id}_{\Gamma'} \otimes \Psi_{\nabla^*})(\nabla'^*\nu \otimes_{\mathcal{A}} \mu) + \nu \otimes_{\mathcal{A}} \nabla^*\mu. \tag{B.8}$$

Fortunately, both procedures lead to the same connection on $\Gamma'^* \otimes_{\mathcal{A}} \Gamma^*$.

Proposition B.2.

$$(\nabla_\otimes)^* = (\nabla^*)_\otimes =: \nabla^*_\otimes. \tag{B.9}$$

Proof. Using

$$\langle \gamma \otimes_{\mathcal{A}} \gamma', (\mathrm{id}_{\Gamma'} \otimes \Psi^*_\nabla)(\nabla'^*\nu \otimes_{\mathcal{A}} \mu)\rangle = \langle \Psi_\nabla(\gamma \otimes_{\mathcal{A}} \langle \gamma', \nabla'^*\nu\rangle), \mu\rangle$$

and

$$\langle \Psi(\gamma \otimes_{\mathcal{A}} \langle \nabla'\gamma', \nu\rangle), \mu\rangle = \langle (\Psi \otimes id_{\Gamma'})(\gamma \otimes_{\mathcal{A}} \nabla'\gamma'), \nu \otimes_{\mathcal{A}} \mu\rangle$$

a direct calculation shows that

$$\langle \gamma \otimes_{\mathcal{A}} \gamma', (\nabla_\otimes)^*(\nu \otimes_{\mathcal{A}} \mu) - (\nabla^*)_\otimes(\nu \otimes_{\mathcal{A}} \mu)\rangle = 0.$$

□

Remark. Our choice among the two possible duals of Γ is related to our use of *left* $\mathcal{A}$-module connections. Let us consider the alternative, the left dual Γ' with contraction $\langle \mu', \gamma \rangle'$†. Then, in the expression $\langle \mu', \nabla\gamma \rangle'$ the 1-form factor of $\nabla\gamma$ cannot be pulled out of the contraction so that there is no (natural) way to define a dual of a left $\mathcal{A}$-module connection. There is an exception, however. In the special case of a linear connection, where $\Gamma = \Omega^1$, we may indeed define a dual connection ∇' on the dual space $\mathcal{X}'$ of Ω^1 via $\langle \nabla' X', \varphi \rangle' = \mathrm{d}\langle X', \varphi \rangle' - \langle X', \nabla\varphi \rangle'$. Then ∇' is a left $\mathcal{A}$-module connection. We emphasized earlier that, for a linear connection $\Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$, the two Ω^1-factors of the target space play very different roles. So ∇' should only be taken seriously if there is a good reason to forget about this fact. Of course, if we consider *right* instead of left $\mathcal{A}$-module connections, the correct contraction should be the primed one.

Appendix C. Coactions and extensions of invariant connections

Let $\mathcal{A}$ be a Hopf algebra with unit $\mathbb{1}$ and coproduct Δ, Γ an $\mathcal{A}$-bimodule and Γ' a left $\mathcal{A}$-module. Γ and Γ' are also assumed to be left $\mathcal{A}$-comodules with coactions‡

$$\begin{aligned} &\Delta_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma \qquad \Delta_\Gamma(\gamma) = \sum_k f_k \otimes \gamma_k \\ &\Delta_{\Gamma'} : \Gamma' \to \mathcal{A} \otimes \Gamma' \qquad \Delta_{\Gamma'}(\gamma') = \sum_l f'_l \otimes \gamma'_l. \end{aligned} \tag{C.1}$$

A left coaction on the tensor product $\Gamma \otimes_{\mathcal{A}} \Gamma'$ is then given by

$$\Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma') = \sum_{k,l} f_k f'_l \otimes \gamma_k \otimes_{\mathcal{A}} \gamma'_l \tag{C.2}$$

(see [7], for example). In the frequently used Sweedler notation [23], this reads

$$\Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma') = \gamma_{(-1)}\gamma'_{(-1)} \otimes \gamma_{(0)} \otimes_{\mathcal{A}} \gamma'_{(0)} \tag{C.3}$$

where $\Delta_\Gamma(\gamma) = \gamma_{(-1)} \otimes \gamma_{(0)}$.

Let (Ω^1, d) be a left-covariant first-order differential calculus on $\mathcal{A}$ and $\nabla : \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ a left $\mathcal{A}$-module connection. ∇ is called *left-invariant* if

$$\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma} \circ \nabla = (\mathrm{id} \otimes \nabla) \circ \Delta_\Gamma \tag{C.4}$$

where $\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma}$ is the left coaction on $\Omega^1 \otimes_{\mathcal{A}} \Gamma$ induced by the left coactions on Ω^1 and Γ. As a consequence of this definition, if $\gamma \in \Gamma$ is left-invariant (i.e. $\Delta_\Gamma(\gamma) = \mathbb{1} \otimes \gamma$) and if also ∇ is left-invariant, then $\nabla\gamma$ is left-invariant, i.e. $\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma} \nabla\gamma = \mathbb{1} \otimes \nabla\gamma$.

Proposition C.1. Let $\mathcal{A}$ be a Hopf algebra, Ω^1 a left-covariant differential calculus on $\mathcal{A}$, Γ an $\mathcal{A}$-bimodule and left $\mathcal{A}$-comodule with a left-invariant extensible connection ∇. Then the associated bimodule homomorphism $\Psi_\nabla : \Gamma \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is also left-invariant, i.e.

$$\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma} \circ \Psi_\nabla = (\mathrm{id} \otimes \Psi_\nabla) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Omega^1}. \tag{C.5}$$

† It still has to be clarified whether the two duals, Γ^* and Γ' are isomorphic in some (natural) sense.
‡ In the following it will be sufficient to consider a left coaction as a linear map $\Gamma \to \mathcal{A} \otimes \Gamma$ such that $\Delta_\Gamma(f\gamma) = \Delta(f)\Delta_\Gamma(\gamma)$. We will need a refinement in proposition C.4 below, see the next footnote.

Proof. Since all maps are $\mathbb{C}$-linear, it suffices to check the invariance condition on elements of the form $\gamma \otimes_{\mathcal{A}} \mathrm{d}f$ with $\gamma \in \Gamma$ and $f \in \mathcal{A}$:

$$\begin{aligned}
\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma} \circ \Psi_{\nabla}(\gamma \otimes_{\mathcal{A}} \mathrm{d}f) &= \Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma}(\nabla(\gamma f) - (\nabla\gamma) f) \\
&= (\mathrm{id} \otimes \nabla) \circ \Delta_{\Gamma}(\gamma f) - ((\mathrm{id} \otimes \nabla) \circ \Delta_{\Gamma}(\gamma))\Delta(f) \\
&= (\mathrm{id} \otimes \nabla)(\Delta_{\Gamma}(\gamma)\Delta(f)) - ((\mathrm{id} \otimes \nabla) \circ \Delta_{\Gamma}(\gamma))\Delta(f) \\
&= \gamma_{(-1)} f_{(1)} \otimes [\nabla(\gamma_{(0)} f_{(2)}) - \nabla(\gamma_{(0)}) f_{(2)}] \\
&= (\mathrm{id} \otimes \Psi_{\nabla}) \circ (\gamma_{(-1)} f_{(1)} \otimes \gamma_{(0)} \otimes_{\mathcal{A}} \mathrm{d}f_{(2)}) \\
&= (\mathrm{id} \otimes \Psi_{\nabla}) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Omega^1}(\gamma \otimes_{\mathcal{A}} \mathrm{d}f).
\end{aligned}$$

□

Proposition C.2. Let $\mathcal{A}$ be a Hopf algebra, Ω^1 a left-covariant differential calculus on $\mathcal{A}$, and Γ, Γ' two $\mathcal{A}$-bimodules which are also left $\mathcal{A}$-comodules. Let ∇, ∇' be left-invariant connections on Γ and Γ', respectively. If ∇ is extensible (with associated bimodule homomorphism Ψ_{∇}), then the product connection $\nabla_{\otimes}$ given by (A.11) is a left-invariant connection on $\Gamma \otimes_{\mathcal{A}} \Gamma'$.

Proof.

$$\begin{aligned}
\Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma\otimes_{\mathcal{A}}\Gamma'} \circ \nabla_{\otimes}(\gamma \otimes_{\mathcal{A}} \gamma') &= \Delta_{(\Omega^1\otimes_{\mathcal{A}}\Gamma)\otimes_{\mathcal{A}}\Gamma'}(\nabla\gamma \otimes_{\mathcal{A}} \gamma') \\
&\quad + \Delta_{(\Omega^1\otimes_{\mathcal{A}}\Gamma)\otimes_{\mathcal{A}}\Gamma'} \circ (\Psi_{\nabla} \otimes \mathrm{id})(\gamma \otimes_{\mathcal{A}} \nabla'\gamma') \\
&= (\mathrm{id} \otimes \nabla \otimes \mathrm{id}) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma') \\
&\quad + (\mathrm{id} \otimes \Psi_{\nabla} \otimes \mathrm{id}) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}(\Omega^1\otimes_{\mathcal{A}}\Gamma')}(\gamma \otimes_{\mathcal{A}} \nabla'\gamma') \\
&= (\mathrm{id} \otimes \nabla \otimes \mathrm{id}) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma') \\
&\quad + (\mathrm{id} \otimes \Psi_{\nabla} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \nabla') \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma') \\
&= (\mathrm{id} \otimes \nabla_{\otimes}) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\gamma \otimes_{\mathcal{A}} \gamma').
\end{aligned}$$

□

Proposition C.3. Let $\mathcal{A}$ be a Hopf algebra and (Ω^1, d) a left-covariant (first-order) differential calculus on $\mathcal{A}$ which is inner, i.e. there is a 1-form ρ such that $\mathrm{d}f = [\rho, f]$ for all $f \in \mathcal{A}$. Let ρ be left-invariant and $\Psi : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ an $\mathcal{A}$-bimodule homomorphism so that

$$\nabla^{\Psi}\varphi = \rho \otimes_{\mathcal{A}} \varphi - \Psi(\varphi \otimes_{\mathcal{A}} \rho) \tag{C.6}$$

defines a linear connection. Then ∇^{Ψ} is left-invariant if and only if Ψ is left-invariant.

Proof. '$\Rightarrow$'. The connection ∇^{Ψ} is extensible and we have $\Psi = \Psi_{\nabla}$. Hence Ψ is left-invariant according to proposition C.2.

'$\Leftarrow$'. If Ψ is left-invariant, then also ∇^{Ψ} since

$$\begin{aligned}
\Delta_{\Omega^1\otimes_{\mathcal{A}}\Omega^1} \circ \nabla^{\Psi}\varphi &= \Delta_{\Omega^1\otimes_{\mathcal{A}}\Omega^1}(\rho \otimes_{\mathcal{A}} \varphi) - (\mathrm{id} \otimes \Psi) \circ \Delta_{\Omega^1\otimes_{\mathcal{A}}\Omega^1}(\varphi \otimes_{\mathcal{A}} \rho) \\
&= \varphi_{(-1)} \otimes \rho \otimes_{\mathcal{A}} \varphi_{(0)} - (\mathrm{id} \otimes \Psi)(\varphi_{(-1)} \otimes \varphi_{(0)} \otimes_{\mathcal{A}} \rho) \\
&= \varphi_{(-1)} \otimes \nabla^{\Psi}\varphi_{(0)} \\
&= (\mathrm{id} \otimes \nabla^{\Psi}) \circ \Delta_{\Omega^1}(\varphi).
\end{aligned}$$

□

Let us now consider two $\mathcal{A}$-bimodules Γ, Γ' with *right* coactions

$$
\begin{aligned}
&{}_{\Gamma}\Delta : \Gamma \to \Gamma \otimes \mathcal{A} \qquad {}_{\Gamma}\Delta(\gamma) = \sum_k \gamma_k \otimes f_k \\
&{}_{\Gamma'}\Delta : \Gamma' \to \Gamma' \otimes \mathcal{A} \qquad {}_{\Gamma'}\Delta(\gamma') = \sum_l \gamma'_l \otimes f'_l .
\end{aligned}
\tag{C.7}
$$

Then

$$
{}_{\Gamma\otimes_{\mathcal{A}}\Gamma'}\Delta(\gamma \otimes_{\mathcal{A}} \gamma') = \sum_{k,l} \gamma_k \otimes_{\mathcal{A}} \gamma'_l \otimes f_k f'_l \tag{C.8}
$$

defines a right coaction on $\Gamma \otimes_{\mathcal{A}} \Gamma'$.

Let (Ω^1, d) be a *right*-covariant first-order differential calculus on $\mathcal{A}$. If an $\mathcal{A}$-bimodule Γ has a right coaction ${}_{\Gamma}\Delta : \Gamma \to \Gamma \otimes \mathcal{A}$, then *right-invariance* of a connection ∇ is defined by

$$
{}_{\Omega^1\otimes_{\mathcal{A}}\Gamma}\Delta \circ \nabla = (\nabla \otimes \mathrm{id}) \circ {}_{\Gamma}\Delta \tag{C.9}
$$

where ${}_{\Omega^1\otimes_{\mathcal{A}}\Gamma}\Delta$ is the right coaction on $\Omega^1 \otimes_{\mathcal{A}} \Gamma$ induced by the right coactions on Ω^1 and Γ. With these notions, the last three propositions in this section remain valid if 'left' is everywhere replaced by 'right' (with the exception that we still consider left $\mathcal{A}$-module connections).

In the following we demonstrate that invariance properties of connections are also transfered to their duals. First we establish the existence of a left coaction on the dual of a bimodule with a left coaction.

Proposition C.4. Let $\mathcal{A}$ be a Hopf algebra and Γ a left-covariant bimodule over $\mathcal{A}$ with coaction Δ_Γ†. Then the dual module Γ^* has a unique left-covariant bimodule structure with coaction $\Delta_{\Gamma^*} : \Gamma^* \to \mathcal{A} \otimes \Gamma^*$ such that

$$
(\mathrm{id} \otimes \langle , \rangle) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma^*} = \Delta \circ \langle , \rangle \tag{C.10}
$$

where $\langle , \rangle$ denotes the contraction mapping $\Gamma \otimes_{\mathcal{A}} \Gamma^* \to \mathcal{A}$.

Proof. According to theorem 2.1 in [7] a left-covariant bimodule Γ has a left $\mathcal{A}$-module basis of left-invariant elements $\{\gamma^i\}$ (where i runs through some index set). Let $\{\mu_j\}$ be the dual basis of Γ^*. Assuming the existence of Δ_{Γ^*}, (C.10) leads to $\mu_{j(-1)} \otimes \langle \gamma^i, \mu_{j(0)} \rangle = \delta^i_j \mathbb{1} \otimes \mathbb{1}$ which implies $\mu_{j(-1)} \in \mathbb{C}$. Now $(\epsilon \otimes \mathrm{id}) \circ \Delta_\Gamma = \mathrm{id}$ shows that the μ_j are left-invariant, i.e. $\Delta_{\Gamma^*}(\mu_j) = \mathbb{1} \otimes \mu_j$. As a consequence, the coaction is unique.

Let us now prove the existence of the coaction on Γ^*. According to theorem 2.1 in [7] there are maps ${F^i}_j : \mathcal{A} \to \mathcal{A}$ such that

$$
\gamma^i f = \sum_k {F^i}_k(f) \gamma^k \qquad \Delta({F^i}_k(f)) = (\mathrm{id} \otimes {F^i}_k)\Delta(f) \qquad \forall f \in \mathcal{A}.
$$

Then

$$
\begin{aligned}
\langle \gamma^i, f\mu_j \rangle = \langle \gamma^i f, \mu_j \rangle &= \sum_k {F^i}_k(f) \langle \gamma^k, \mu_j \rangle = {F^i}_j(f) \\
&= \sum_k \langle \gamma^i, \mu_k \rangle {F^k}_j(f) = \langle \gamma^i, \sum_k \mu_k {F^k}_j(f) \rangle
\end{aligned}
$$

† A left-covariant bimodule over a Hopf algebra $\mathcal{A}$ is an $\mathcal{A}$-bimodule Γ together with a map (coaction) $\Delta_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma$ such that $\Delta_\Gamma(f\gamma f') = \Delta(f)\Delta_\Gamma(\gamma)\Delta(f')$ for all $f, f' \in \mathcal{A}, \gamma \in \Gamma$. It has to satisfy the equations $(\Delta \otimes \mathrm{id}) \circ \Delta_\Gamma = (\mathrm{id} \otimes \Delta_\Gamma) \circ \Delta_\Gamma$ and $(\epsilon \otimes \mathrm{id}) \circ \Delta_\Gamma = \mathrm{id}$ where ϵ is the counit. See also [7].

implies $f\mu_j = \sum_k \mu_k F^k{}_j(f)$. Now we define the coaction Δ_{Γ^*} on the basis μ_j by $\Delta_{\Gamma^*}(\mu_j) := \mathbb{1} \otimes \mu_j$ and extend it via $\Delta_{\Gamma^*}(\mu_j f) := \Delta_{\Gamma^*}(\mu_j)\Delta(f)$ for all $f \in \mathcal{A}$. Then

$$\begin{aligned}
\Delta_{\Gamma^*}(f\mu_j) &= \Delta_{\Gamma^*}\left(\sum_k \mu_k F^k{}_j(f)\right) \\
&= \sum_k \Delta_{\Gamma^*}(\mu_k)\Delta(F^k{}_j(f)) \\
&= \sum_k (\mathbb{1} \otimes \mu_k)(id \otimes F^k{}_j)\Delta(f) \\
&= \sum_k f_{(1)} \otimes \mu_k F^k{}_j(f_{(2)}) \\
&= \sum_k f_{(1)} \otimes f_{(2)}\mu_j \\
&= \Delta(f)\Delta_\Gamma(\mu_j).
\end{aligned}$$

It is now sufficient to verify the remaining defining properties of a left-covariant bimodule on the left-invariant basis elements μ_j, and furthermore (C.10) on $\{\gamma^i\}$ and $\{\mu_j\}$. We leave this to the reader. $\square$

After some preparations in the following Lemma, we prove that left-invariance of a connection on a left-covariant bimodule translates to invariance of the dual connection which lives on the dual left-covariant bimodule.

Lemma C.1.

$$(\mathrm{id} \otimes \langle , \rangle) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma^*\otimes_{\mathcal{A}}\Omega^1}(\gamma \otimes_{\mathcal{A}} \hat{\mu}) = \Delta_{\Omega^1}\langle \gamma, \hat{\mu}\rangle \qquad \forall \hat{\mu} \in \Gamma^* \otimes_{\mathcal{A}} \Omega^1 \tag{C.11}$$

$$(\mathrm{id} \otimes \langle , \rangle) \circ \Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma\otimes_{\mathcal{A}}\Gamma^*}(\hat{\gamma} \otimes_{\mathcal{A}} \mu) = \Delta_{\Omega^1}\langle \hat{\gamma}, \mu\rangle \qquad \forall \hat{\gamma} \in \Omega^1 \otimes_{\mathcal{A}} \Gamma. \tag{C.12}$$

Proof. This is a straightforward calculation using (C.10). $\square$

Proposition C.5. Let $\mathcal{A}$ be a Hopf algebra, Γ a left-covariant bimodule over $\mathcal{A}$ and (Ω^1, d) a left-covariant first-order differential calculus on $\mathcal{A}$. If $\nabla : \Gamma \to \Omega^1 \otimes_{\mathcal{A}} \Gamma$ is left-invariant, then the dual connection $\nabla^* : \Gamma^* \to \Gamma^* \otimes_{\mathcal{A}} \Omega^1$ is also left-invariant.

Proof. We have to show that

$$\Delta_{\Gamma^*\otimes_{\mathcal{A}}\Omega^1} \circ \nabla^* = (\mathrm{id} \otimes \nabla^*) \circ \Delta_{\Gamma^*}.$$

Let $\{\gamma^i\}$ be a left-invariant left $\mathcal{A}$-module basis of Γ [7]. We introduce mappings $C^i : \Gamma^* \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1, \mu \otimes_{\mathcal{A}} \varphi \mapsto \langle \gamma^i, \mu\rangle\varphi$. Then

$$\begin{aligned}
(\mathrm{id} \otimes C^i)(\mathrm{id} \otimes \nabla^*)\Delta_{\Gamma^*}(\mu) &= \mu_{(-1)} \otimes \langle \gamma^i, \nabla^*\mu_{(0)}\rangle \\
&= \mu_{(-1)} \otimes [\mathrm{d}\langle \gamma^i, \mu_{(0)}\rangle - \langle \nabla\gamma^i, \mu_{(0)}\rangle] \\
&= [(\mathrm{id} \otimes \mathrm{d}) \circ (\mathrm{id} \otimes \langle , \rangle) - (\mathrm{id} \otimes \langle , \rangle) \circ (\mathrm{id} \otimes \nabla \otimes \mathrm{id})] \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma^*}(\gamma^i \otimes_{\mathcal{A}} \mu) \\
&= (\mathrm{id} \otimes \mathrm{d}) \circ \Delta\langle \gamma^i, \mu\rangle - (\mathrm{id} \otimes \langle , \rangle) \circ \Delta_{\Omega^1\otimes_{\mathcal{A}}\Gamma\otimes_{\mathcal{A}}\Gamma^*} \circ (\nabla \otimes \mathrm{id})(\gamma^i \otimes_{\mathcal{A}} \mu) \\
&= \Delta_{\Omega^1}(\mathrm{d}\langle \gamma^i, \mu\rangle - \langle \nabla\gamma^i, \mu\rangle) \\
&= \Delta_{\Omega^1}\langle \gamma^i, \nabla^*\mu\rangle \\
&= (\mathrm{id} \otimes \langle , \rangle) \circ \Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma^*\otimes_{\mathcal{A}}\Omega^1}(\gamma^i \otimes_{\mathcal{A}} \nabla^*\mu) \\
&= (\nabla^*\mu)_{(-1)} \otimes \langle \gamma^i, (\nabla^*\mu)_{(0)}\rangle \\
&= (\mathrm{id} \otimes C^i) \circ \Delta_{\Gamma^*\otimes_{\mathcal{A}}\Omega^1}(\nabla^*\mu)
\end{aligned}$$

using the left-invariance of $\nabla\gamma^i$ and lemma C.1 It remains to show that $(\mathrm{id}\otimes C^i)(\xi) = 0$ for all i implies $\xi = 0$ (where $\xi \in \mathcal{A} \otimes \Gamma^* \otimes_{\mathcal{A}} \Omega^1$). ξ has an expression

$$\xi = \sum_{\alpha,j,r} f_\alpha \otimes \mu_j \otimes_{\mathcal{A}} \varphi_r \xi_{\alpha jr}$$

with $\xi_{\alpha jr} \in \mathcal{A}$. Here $\{\mu_j\}$ is the basis of Γ^* dual to $\{\gamma^i\}$ and $\{\varphi_r\}$ is a right $\mathcal{A}$-module basis of Ω^1. Evaluation of $(\mathrm{id}\otimes C^i)(\xi) = 0$ now leads to $\sum_\alpha f_\alpha \otimes \xi_{\alpha ir} = 0$ for all i and all r. Hence $\xi = 0$. □

Corresponding results are obtained for a right-covariant bimodule Γ with coaction ${}_\Gamma\Delta$ and right-invariant connections on it. In this case (C.10) is replaced by

$$(\langle,\rangle \otimes \mathrm{id}) \circ {}_{\Gamma\otimes_{\mathcal{A}}\Gamma^*}\Delta = \Delta \circ \langle,\rangle. \tag{C.13}$$

Appendix D. Two-sided connections

The problem of extensibility of a connection discussed in appendix A disappears if we modify its definition as follows.

Definition. A *two-sided connection*† on an $\mathcal{A}$-bimodule Γ is a map $\nabla : \Gamma \to (\Omega^1 \otimes_{\mathcal{A}} \Gamma) \oplus (\Gamma \otimes_{\mathcal{A}} \Omega^1)$ such that

$$\nabla(f\gamma f') = \mathrm{d}f \otimes_{\mathcal{A}} \gamma f' + f\gamma \otimes_{\mathcal{A}} \mathrm{d}f' + f(\nabla\gamma)f' \tag{D.1}$$

for all $f, f' \in \mathcal{A}$ and $\gamma \in \Gamma$.

The difference of two such connections is a bimodule homomorphism. The following example demonstrates that the concept of a two-sided connection is much more restrictive than that of the usual one.

Example. For a first-order differential calculus which is inner, i.e. there is a 1-form ρ such that $\mathrm{d}f = [\rho, f]$ for all $f \in \mathcal{A}$,

$$\nabla\varphi := \rho \otimes_{\mathcal{A}} \varphi - \varphi \otimes_{\mathcal{A}} \rho \tag{D.2}$$

defines a two-sided linear connection. In the particular case of the three-dimensional bicovariant differential calculus on $\mathcal{S}_3$, we observed in section 6 that there is no non-trivial bimodule homomorphism $\Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$. Hence, the two-sided connection defined above is the only one in this case.

A two-sided connection extends to a map $\Omega \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Omega \to \Omega \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Omega$ via

$$\nabla(\varphi\gamma\varphi') = (\mathrm{d}\varphi)\gamma\varphi' + (-1)^r(\nabla\gamma)\varphi' + (-1)^{r+s}\varphi\gamma\,\mathrm{d}\varphi' \tag{D.3}$$

where $\varphi \in \Omega^r$ and $\gamma \in \bigoplus_{k=0}^{s} \Omega^k \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Omega^{s-k}$. The curvature of ∇ then turns out to be an $\mathcal{A}$-bimodule homomorphism, i.e.

$$\nabla^2(f\gamma f') = f(\nabla^2\gamma)f' \qquad \forall f,\ f' \in \mathcal{A},\ \gamma \in \Gamma \tag{D.4}$$

a nice property not shared, in general, by ordinary connections.

† See also [21, 24] for related structures.

Appendix E. Invariant tensor fields on a finite group

From two $\mathcal{A}$-bimodules Γ and Γ' we can build the tensor product $\Gamma \otimes_{\mathcal{A}} \Gamma'$. If both modules carry a (left or right) $\mathcal{A}$-comodule structure, there is a comodule structure on the tensor product space (see appendix C). In case of left comodules, the left-invariance condition for a tensor field $\alpha \in \Gamma \otimes_{\mathcal{A}} \Gamma'$ reads $\Delta_{\Gamma\otimes_{\mathcal{A}}\Gamma'}(\alpha) = \mathbb{1}\otimes\alpha$. For right comodules this is replaced by the right-invariance condition ${}_{\Gamma\otimes_{\mathcal{A}}\Gamma'}\Delta(\alpha) = \alpha\otimes\mathbb{1}$. In the following we consider a bicovariant (first-order) differential calculus on a finite group G. Besides being an $\mathcal{A}$-bimodule, the space Ω^1 is then a left and right $\mathcal{A}$-comodule. Each tensor field $\alpha \in \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ can be written as

$$\alpha = \alpha_{\underline{g},\underline{g}'}\theta^{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g}'} \tag{E.1}$$

where summations run over the set $\hat{G} = \{g \in G | \theta^g \neq 0\}$. Left-invariance of α then means $\alpha_{g,g'} \in \mathbb{C}$. Bi-invariance leads to the additional condition

$$\alpha_{\underline{g},\underline{g}'}\theta^{\mathrm{ad}(h)\underline{g}} \otimes_{\mathcal{A}} \theta^{\mathrm{ad}(h)\underline{g}'} = \alpha_{\underline{g},\underline{g}'}\theta^{\underline{g}} \otimes_{\mathcal{A}} \theta^{\underline{g}'} \qquad \forall h \in G. \tag{E.2}$$

For fixed $h \in G$, the map $\mathrm{ad}(h) : G \to G$ is a bijection. Hence

$$\alpha_{\underline{g},\underline{g}'}\theta^{\mathrm{ad}(h)\underline{g}} \otimes_{\mathcal{A}} \theta^{\mathrm{ad}(h)\underline{g}'} = \alpha_{\mathrm{ad}(h^{-1})\underline{k},\mathrm{ad}(h^{-1})\underline{k}'}\theta^{\underline{k}} \otimes_{\mathcal{A}} \theta^{\underline{k}'} \tag{E.3}$$

and the bi-invariance condition becomes

$$\alpha_{\mathrm{ad}(h)g,\mathrm{ad}(h)g'} = \alpha_{g,g'} \in \mathbb{C} \qquad \forall g, g' \in \hat{G}, h \in G. \tag{E.4}$$

For a bicovariant differential calculus with bimodule isomorphism σ, the condition for a tensor field α to be s-symmetric is

$$\alpha_{g,g'} = \alpha_{g',\mathrm{ad}(g')g} \qquad \forall g, g' \in \hat{G}. \tag{E.5}$$

α is s-antisymmetric iff

$$\alpha_{g,g'} = -\alpha_{g',\mathrm{ad}(g')g} \qquad \forall g, g' \in \hat{G}. \tag{E.6}$$

Example. Let us consider $\mathcal{S}_3$ with the universal (first-order) differential calculus (see section 6). In matrix notation, the coefficients of an s-symmetric tensor field α, as given by (E.1), must have the form

$$(\alpha_{g,g'}) = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 & \beta_1 & \beta_2 \\ \alpha_5 & \alpha_2 & \alpha_4 & \beta_3 & \beta_1 \\ \alpha_4 & \alpha_5 & \alpha_3 & \beta_2 & \beta_3 \\ \beta_2 & \beta_1 & \beta_3 & \gamma_1 & \gamma_3 \\ \beta_1 & \beta_3 & \beta_2 & \gamma_3 & \gamma_2 \end{pmatrix} \tag{E.7}$$

where the entries are (arbitrary) elements of $\mathcal{A}$ (respectively constants if α is left-invariant). Rows and columns are arranged, respectively, according to the index sequence $\{a, b, c, ab, ba\}$. For an s-antisymmetric tensor field we obtain

$$(\alpha_{g,g'}) = \begin{pmatrix} 0 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & \beta_3 & \beta_1 \\ 0 & 0 & 0 & \beta_2 & \beta_3 \\ -\beta_2 & -\beta_1 & -\beta_3 & 0 & \gamma \\ -\beta_1 & -\beta_3 & -\beta_2 & -\gamma & 0 \end{pmatrix}. \tag{E.8}$$

For a w-symmetric tensor field we find

$$(\alpha_{g,g'}) = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 & \beta_1 & \beta_1' \\ \alpha_7 & \alpha_2 & \alpha_6 & \beta_1'' & \beta_3 \\ \alpha_8 & \alpha_9 & \alpha_3 & \beta_3' & \beta_3'' \\ \beta_1' - \beta_2' + \beta_3' & \beta_2 & \beta_2'' & \gamma_1 & \gamma_3 \\ \beta_1 - \beta_2 + \beta_3 & \beta_1'' - \beta_2'' + \beta_3'' & \beta_2' & \gamma_3 & \gamma_2 \end{pmatrix} \tag{E.9}$$

and the coefficients of a w-antisymmetric tensor field are given by

$$(\alpha_{g,g'}) = \begin{pmatrix} 0 & \alpha_1 & -\alpha_3-\alpha_4 & \beta_1 & \beta_1' \\ \alpha_3 & 0 & \alpha_2 & \beta_1'' & \beta_3 \\ -\alpha_1-\alpha_2 & \alpha_4 & 0 & \beta_3' & \beta_3'' \\ -\beta_1'-\beta_2'-\beta_3' & \beta_2 & \beta_2'' & 0 & \gamma \\ -\beta_1-\beta_2-\beta_3 & -\beta_1''-\beta_2''-\beta_3'' & \beta_2' & -\gamma & 0 \end{pmatrix}. \tag{E.10}$$

α is bi-invariant iff the coefficient matrix has the form

$$(\alpha_{g,g'}) = \begin{pmatrix} \alpha & \alpha' & \alpha' & \beta & \beta \\ \alpha' & \alpha & \alpha' & \beta & \beta \\ \alpha' & \alpha' & \alpha & \beta & \beta \\ \beta & \beta & \beta & \gamma & \gamma' \\ \beta & \beta & \beta & \gamma' & \gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \beta' & \beta' \\ 0 & 0 & 0 & \beta' & \beta' \\ 0 & 0 & 0 & \beta' & \beta' \\ -\beta' & -\beta' & -\beta' & 0 & 0 \\ -\beta' & -\beta' & -\beta' & 0 & 0 \end{pmatrix} \tag{E.11}$$

with constants $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$. As expressed above, it turns out to be a sum of s-symmetric and s-antisymmetric tensors.

Appendix F. Finite group actions on a finite set

Within the framework of non-commutative geometry of finite sets one can also formulate the notion of covariance with respect to a group action on a finite set. Let $M = \{x, y, \ldots\}$ be this set and G a finite group acting on M from the left,

$$G \times M \to M \qquad (g, x) \mapsto g \cdot x. \tag{F.1}$$

For $g, g' \in G$ and $x \in M$ we have $(gg') \cdot x = g \cdot (g' \cdot x)$. The action of the neutral element $e \in G$ is trivial, i.e. $e \cdot x = x$ for all $x \in M$. We denote the algebra of $\mathbb{C}$-valued functions on M and G by $\mathcal{H}$ and $\mathcal{A}$, respectively. $\mathcal{A}$ is a Hopf algebra over $\mathbb{C}$. The group action induces a *left coaction* $\Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ via

$$\Delta_{\mathcal{H}}(f)(g, x) = f(g \cdot x). \tag{F.2}$$

Since $\Delta_{\mathcal{H}}$ is compatible with the multiplication in $\mathcal{H}$, the latter is turned into a (left) $\mathcal{A}$-*comodule algebra*. In particular,

$$\Delta_{\mathcal{H}}(e_x) = \sum_{g \in G} e_g \otimes e_{g^{-1} \cdot x}. \tag{F.3}$$

A (first-order) differential calculus on M (or $\mathcal{H}$) with space of 1-forms Ω^1 is called G-*covariant* iff there is a linear map $\Delta_{\Omega^1} : \Omega^1 \to \mathcal{A} \otimes \Omega^1$ such that

$$\Delta_{\Omega^1}(f \varphi h) = \Delta_{\mathcal{H}}(f) \Delta_{\Omega^1}(\varphi) \Delta_{\mathcal{H}}(h) \qquad \forall f \in \mathcal{A}, h \in \mathcal{H} \tag{F.4}$$

and

$$\Delta_{\Omega^1} \circ \mathrm{d} = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta_{\mathcal{H}}. \tag{F.5}$$

As a consequence,

$$\Delta_{\Omega^1}(e_{x,y}) = \sum_{g \in G} e_{g^{-1}} \otimes e_{g \cdot x, g \cdot y}. \tag{F.6}$$

We obtain all G-covariant differential calculi on M by deleting sets of arrows from the universal graph (the digraph which corresponds to the universal differential calculus on M). These correspond to G-orbits in $(M \times M)'$. A (non-trivial) G-covariant differential calculus is called *irreducible* if it belongs to a single orbit. All (non-trivial) differential calculi are then obtained as unions of irreducible ones.

Example 1. Let M be a finite set with n elements and $G = \mathcal{S}_n$, the symmetric group. Obviously, the action of $\mathcal{S}_n$ on $(M \times M)'$ is transitive, i.e. all (x, y) where $x \neq y$ belong to the same $\mathcal{S}_n$-orbit. Therefore, the only $\mathcal{S}_n$-covariant (first-order) differential calculi on M are the universal and the trivial one.

Example 2. Instead of the action of the whole symmetric group we may consider actions of subgroups of $\mathcal{S}_n$. For $n = 3$, for example, we have the (non-trivial) subgroups

$$G_1 = \{e, a\} \qquad G_2 = \{e, b\} \qquad G_3 = \{e, c\} \qquad G_4 = \{e, ab, ba\}. \tag{F.7}$$

Denoting the points of M by 1, 2, 3, we can calculate the orbits in $(M \times M)'$. For the action of G_1, we obtain

$$\mathcal{O}_1 = \{(1, 2), (2, 1)\} \qquad \mathcal{O}_2 = \{(1, 3), (2, 3)\} \qquad \mathcal{O}_3 = \{(3, 1), (3, 2)\}. \tag{F.8}$$

The graphs which determine the irreducible calculi are depicted in figure F1.

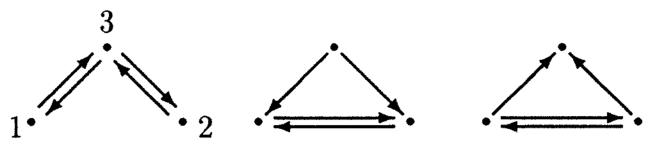

Figure F1. The digraphs coresponding to the irreducible G_1-covariant first-order differential calculi on a 3-point set.

In the case of G_4 acting on M we obtain

$$\mathcal{O}_1 = \{(1, 2), (2, 3), (3, 1)\} \qquad \mathcal{O}_2 = \{(1, 3), (2, 1), (3, 2)\}. \tag{F.9}$$

The graphs corresponding to irreducible calculi are displayed in figure F2.

Figure F2. The digraphs corresponding to the irreducible G_4-covariant first-order differential calculi on a 3-point set.

Of course, one can proceed with the formalism by defining invariance of tensors and connections on M. All this will be explored in detail in a separate work.

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